Do six of the following problems.

1. (a) Define what it means for a collection $\mathcal{F}$ to be an algebra.
   
   (b) Define what it means for a collection $\mathcal{F}$ to be a $\sigma$-algebra.

   (c) Give an example of an algebra that is not a $\sigma$-algebra.

2. (a) Define what it means for a set $A \subset \mathbb{R}$ to be Lebesgue measurable.

   (b) Show that the interval $(0, 1)$ is Lebesgue measurable.

3. Show that if $f \in L^1(A)$ and for any measurable set $B \subset A$
   $$\int_B f(x)dx = 0$$
   then $f(x) = 0$ almost everywhere on $A$.

4. Let $E$ be a measurable set with finite measure and let $\{f_n\}$ be a sequence
   of nonnegative integrable functions satisfying $\int_E f_n \to 0$.

   (a) Show that the sequence $\{f_n\}$ converges to 0 in measure.

   (b) Give an example that shows that $\{f_n\}$ need not converge to 0 almost
   everywhere.

5. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function and define the
   function $F : [0, 1] \to \mathbb{R}$ by $F(t) = \int_0^t f(x) \sin(xt) dx$.

   (a) Show that $F$ is well defined.

   (b) Show that $F$ is differentiable and $F'(t) = \int_0^1 xf(x) \cos(xt) dx$. 
(c) Show that $F$ has derivatives of all orders and that

$$F^{(2n)}(t) = (-1)^n \int_0^1 x^{2n} f(x) \sin(xt) \, dx$$

and

$$F^{(2n-1)}(t) = (-1)^{n+1} \int_0^1 x^{2n-1} f(x) \cos(xt) \, dx.$$  

6. Let $f$ be a bounded measurable function on $[0,1]$. Prove or disprove:

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$  

7. Let $M$ be a set and $\rho$ a function $\rho : M \times M \to \mathbb{R}$.

   (a) Define what it means that a couple $(M, \rho)$ is a complete metric space.

   (b) Let $\rho(x, y) = |\arctan x - \arctan y|$. Prove or disprove: $(\mathbb{R}, \rho)$ is a complete metric space.

8. Let $V$ be a vector space, let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two inner products on $V$, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the norms corresponding to these inner products. Suppose that $V$ is complete in both norms and that it is separable in each of the norm-toplogies. Show that the normed spaces $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are isomorphic.

9. Let $m$ denote the Lebesgue measure on $\mathbb{R}$ and let $\nu$ be a measure defined as $\nu(A) = m(A \cap [-2,2])$ for every Lebesgue measurable set $A$. Prove that $\nu$ is absolutely continuous with respect to $m$ and find explicitly the Radon-Nikodym derivative $d\nu/dm$. 

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