February 28, 2003

ANALYSIS PRELIMINARY EXAMINATION

(ANSWER ANY FIVE OF THE FOLLOWING EIGHT QUESTIONS)

1. Let $x_n$ be a bounded sequence of elements in a separable Hilbert space $H$. Show that there exists a subsequence $x_{n_k}$ which converges weakly.

2. (a) Give the definition of a measurable function $f : \mathbb{R} \to \mathbb{R}$. (b) Let $f_n$ be a sequence of measurable functions on a measurable set $E$. Prove or disprove: the function $f = \lim \sup_{n \to \infty} f_n$ is measurable.

3. Let $f_n$ be a sequence of Lebesgue measurable functions on $[0, 1]$.
   (a) Define convergence in $L^p$ and in measure for $f_n$ to a function $f$ on $[0, 1]$.
   (b) Prove or disprove: The sequence $f_n$ converges in $L^p$, $p \geq 1$, to $f$ implies that $f_n$ converges in measure to $f$.

4. Let $(X, \mu)$ be a measure space, $f \in L^1(X, \mu)$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set $E$ satisfying $\mu(E) < \delta$ the following holds
   \[ \int_E |f|d\mu < \varepsilon \]

5. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of measurable functions defined on $E \subset \mathbb{R}$ such that $\mu(E) < \infty$. Suppose that $f_n$ converges to $f$ in measure and $g_n$ converges to $g$ in measure. Prove or disprove: $f_ng_n$ converges in measure to $fg$. Consider the same problem when $\mu(E) = \infty$.

6. Let $f$ be a continuous real-valued function, and let $g$ be a (Lebesgue) measurable real-valued function, both defined on $\mathbb{R}$. Prove or disprove: the function $h(x) = g(f(x))$ is (Lebesgue) measurable.

7. Compute $\lim_{n \to \infty} \int_{[0,\infty]} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$.

8. If $f_n$ is a sequence in $L^1(X, \mu)$ which converges uniformly on $X$ to a function $f$, and if $\mu(X) < +\infty$, then
   \[ \int f d\mu = \lim_{n \to \infty} \int f_n d\mu \]
7) Let $1 \leq p \leq \infty$. Suppose that $\{f_n\}$ is a sequence in $L^p[0,1]$ and that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

(a) Show that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e.

(b) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. Show that $f \in L^p[0,1]$ and that $\|f\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p$.

8) Prove or disprove that a finite valued linear functional on a normed space $X$ is discontinuous if and only if $F(\{x \in X : \|x - a\| < r\}) = \mathbb{R}$ for any $a \in X$ and any $r > 0$. 