Analysis Comprehensive Examination

March 4, 2004

Do six of the following eight problems.

1) Let $E$ be a Lebesgue measurable set and let $f_n$ be a sequence of Lebesgue measurable functions. Prove or disprove the following:

(a) If $f_n$ converges to 0 in measure, then
$$
\int_E \frac{|f_n|}{1 + |f_n|} \, d\mu \to 0.
$$

(b) If
$$
\int_E \frac{|f_n|}{1 + |f_n|} \, d\mu \to 0,
$$
then $f_n$ converges to 0 in measure on $E$.

2) Given an $\alpha > 0$ a function $f : (a, b) \to \mathbb{R}$ is Lipschitz of order $\alpha$ if there is a $C > 0$ such that
$$
|f(x) - f(y)| \leq C|x - y|^\alpha
$$
for all $x, y \in (a, b)$.

(a) Assume that $f$ is Lipschitz of order $\alpha > 1$. Prove that $f'(x) = 0$ for all $x \in (a, b)$.

(b) Prove that if $f$ is Lipschitz of order 1 on $(a, b)$, then $f'$ exists almost everywhere and is uniformly bounded.

3) Let $A$ denote the set of all $x = \{\psi_n\}$ in $\ell_2$ such that $\sum_{n=1}^{\infty} \psi_n = 0$. Prove or disprove that $A$ is dense in $\ell_2$.

4) Let $X$ be a real Banach space.

(a) Show that any sequence $\{x_n^*\}$ that converges strongly in $X^*$ converges in the weak* topology.

(b) Show that the weak* topology is Hausdorff.

(c) Show that any weakly convergent sequence $\{x_n^*\}$ in $X^*$ is weak* convergent in $X^*$.

5) Let $m^*$ be Lebesgue outer measure. Prove that a bounded set $E$ is Lebesgue measurable if and only if for every $\varepsilon > 0$ there is a a closed subset $F$ of $E$ such that $m^*(E - F) < \varepsilon$.

6) Consider a positive measure space $(X, \mu)$.

(a) State and prove Fatou’s Lemma.

(b) Does this theorem hold for general measurable functions? Explain.
7) Let $1 \leq p \leq \infty$. Suppose that $\{f_n\}$ is a sequence in $L^p[0,1]$ and that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

(a) Show that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e.

(b) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. Show that $f \in L^p[0,1]$ and that $\|f\|_p = \sum_{n=1}^{\infty} \|f_n\|_p$.

8) Prove or disprove that a bounded linear functional on a normed space $X$ is discontinuous if and only if $F(\{x \in X : \|x - a\| < r\}) = \mathbb{R}$ for any $a \in X$ and any $r > 0$. 