WMU Analysis Comprehensive Exam
Spring 2009

Do six of the following problems.

1) Let \( \mu \) and \( \nu \) be two measures on \( X \). Show that \( \mu + \nu \) is a measure on \( X \).

2) Let \((X, \mathcal{M}, \mu)\) be a measure space and let \( A \) be a dense set in \( \mathbb{R} \). Show that a function \( f \) on \( X \) is measurable if and only if for each \( a \in A \) the set \( f^{-1}([a, \infty)) \) is measurable.

3) Show that a linear functional \( L \) on a normed space is continuous if and only if \( L^{-1}(0) \) is closed.

4) Let \((X, \mathcal{M}, \mu)\) be a complete finite measure space and let \( f \) be a nonnegative measurable function on \((X, \mathcal{M}, \mu)\). Consider \( X \times \mathbb{R} \) with the product measure of \( \mu \) with Lebesgue measure.
   (a) Show that the set \( M = \{(x, \alpha) | 0 \leq \alpha \leq f(x)\} \) is measurable.
   (b) Relate the measure of \( M \) to the integral of \( f \) on \( X \).

5) Let \( f \) be an integrable function that is positive almost everywhere. If \( A \) is a set such that \( \int_A f \, dm = 0 \), then \( m(A) = 0 \).

6) Consider a measure space \((Y, \mathcal{Y}, \nu)\) such that for all \( A \in \mathcal{Y} \), \( \nu(A) \in \{a_1, \ldots, a_k\} \) for some integer \( k \).
   (a) Describe the measurable sets if there are exactly four values for \( \nu(A) \).
   (b) Can there be exactly six values for \( \nu(A) \)? Explain your answer.

7) Show that \( L^\infty([0, 1]) \) with Lebesgue measure is not separable.

8) Demonstrate that if a measure space \((X, \mathcal{B}, \mu)\) is not \( \sigma \)-finite, then the Radon-Nikodym theorem does not hold.