1. Let $A, B$ be any sets with power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$, let $\mathcal{A} \subset \mathcal{P}(A)$ and $\mathcal{B} \subset \mathcal{P}(B)$, and let $f : A \to B$. Define $f(\mathcal{A}) = \{ f(X) : X \in \mathcal{A} \}$ and $f^{-1}(\mathcal{B}) = \{ f^{-1}(Y) : Y \in \mathcal{B} \}$.

(a) If $\mathcal{B}$ is a $\sigma$-algebra, prove that $f^{-1}(\mathcal{B})$ is also a $\sigma$-algebra.

(b) Give an example to show that if $\mathcal{A}$ is a $\sigma$-algebra, $f(\mathcal{A})$ need not be a $\sigma$-algebra.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $g : [a, b] \to \mathbb{R}$ be absolutely continuous.

(a) Show that the function

$$F(x) := \int_0^{g(x)} f(t) dt$$

is absolutely continuous.

(b) Find

$$\frac{d}{dx} F(x)$$

for almost all $x \in [a, b]$.

3. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ be a measurable function.

(a) Show that if for each natural $n$ function $f^n$ is integrable and $\lim_{n \to \infty} \int f^n d\mu$ exists then $|f(x)| \leq 1$ for almost all $x \in X$.

(b) If $f^n$ is integrable for each $n$ and $\int f^n d\mu = c$ for some constant $c$ then show that $f(x) = \chi_A(x)$ for some measurable set $A \subset X$. 


4. (a) State the Baire Category Theorem.

(b) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a sequence of lower semicontinuous functions.
Suppose that, for each \( x \in \mathbb{R}^n \), there exists a constant \( M_x \) such that \( f_n(x) \leq M_x \) for all \( n \).
Prove that there is a nonempty open set \( O \subset \mathbb{R}^n \) and a constant \( M \) such that, for all \( n \),
\[
f_n(x) \leq M, \quad \forall \ x \in O.
\]

5. Let \( X, Y \) be Banach spaces and let \( T : X \to Y \) be a bounded linear operator. If a sequence \( \{x_n\} \subset X \) is weakly convergent, prove that the same is true for \( \{Tx_n\} \).

6. Let \( S \) be the set of all step functions on \([0, 1]\) with rational range and rational partition points.

(a) Show that the closure of \( S \) in \( L_\infty[0, 1] \) contains \( C[0, 1] \).

(b) Show that the \( S \) is dense in \( L_1[0, 1] \).

7. An ordering on a topological space is an order compatible with the topology if when \( x < y \), there are disjoint neighborhoods \( A \) of \( x \) and \( B \) of \( y \) such that for all \( z \in A \) one has \( z < y \) and for all \( w \in B \) one has \( x < w \).

(a) Show that the unit circle cannot be well ordered. (Hint: One proof follows the lines of the proof of the intermediate value theorem.)

(b) Show that you cannot find a well ordering of the plane \( \mathbb{R}^2 \) that is compatible with the Euclidean distance in the plane.