Algebra Preliminary Exam
August 16, 2001

Instructions: Do all seven problems. You will have four hours for this exam.

1. (a) Give statements of the (first) three isomorphism theorems for groups.
   (b) (Goursat’s Lemma) Let $G_1$ and $G_2$ be groups, and let $H$ be a subgroup of $G_1 \times G_2$ such that the two projections $p_1 : H \to G_1$ are surjective. Let $N_1$ be the kernel of $p_2$, and let $N_2$ be the kernel of $p_1$. One can identify $N_1$ and $N_2$ with normal subgroups of $G_1$ and $G_2$ respectively (check!). Prove that $G_1/N_1 \cong G_2/N_2$.

2. Let $G$ be a simple group of order 60. Show that the action of $G$ by conjugation on its set of Sylow subgroups gives an embedding $G \hookrightarrow A_6$ of $G$ into the alternating group on 6 letters.

3. Let $R$ be a ring and $M$ a (left) $R$-module.
   (a) Prove that $\text{End}_R(M) := \text{Hom}_R(M, M)$ is a ring.
   (b) (Schur’s Lemma) Suppose $M$ is a simple $R$-module (recall that $M$ is simple if it is nonzero and has no nontrivial submodules). Prove that $\text{End}_R(M)$ is a division ring.

4. Observe that $\mathbb{Q}$ under addition is a module over the integers $\mathbb{Z}$.
   (a) Show $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.
   (b) Show $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

5. Let $R$ be an integral domain.
   (a) Define what it means for a nonzero element $p$ of $R$ to be prime, and what it means for $p$ to be irreducible.
   (b) Mark the following statements TRUE or FALSE.
      (i) If $p$ is prime then $p$ is irreducible.
      (ii) If $p$ is irreducible then $p$ is prime.
   (c) Of the items in (b), prove those which you marked TRUE.
   (d) Complete the following statement:
      An ideal $P$ in $R$ is prime if and only if $R/P$ is .
      (Recall that, by assumption, a prime ideal is proper.)
   (e) Prove your assertion in (d).
   (f) Show that if $R$ is a principal ideal domain, then $p \in R$ is prime if and only if it is irreducible.

6. Let $K/F$ be a finite separable field extension of prime degree $p$. Suppose $K = F(\theta)$.
   (a) Show that $\theta$ has exactly $p$ distinct Galois conjugates in some algebraic closure of $K$.
   (b) Let $\theta = \theta_1, \theta_2, \ldots, \theta_p$ be the Galois conjugates of $\theta$. Show that if $\theta_2 \in K$ then $K$ is Galois, with cyclic Galois group.

7. Let $\zeta$ be a primitive 12th root of unity.
   (a) Find all the automorphisms of the field extension $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (b) Find the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (c) Find all intermediary field extensions between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$.