1. Let $M$ be a monoid. (Recall that a monoid $M$ is a (non-empty) set, closed under an associative binary operation which has a two-sided identity.)
   (a) Let $F(M)$ be the set of all functions from $M$ into $M$. Prove that $F(M)$ is a monoid under composition of functions.
   (b) Prove that $M$ acts on $M$ by right multiplication.
   (c) Prove that $M$ is isomorphic to a submonoid of $F(M)$.
   (d) Prove that $M$ is a group if and only if $M$ acts transitively on $M$ by right multiplication.

2. Let $P$ be a nontrivial, finite $p$-group, with center $Z(P)$. If $N$ is a nontrivial, normal subgroup of $P$, prove that $N \cap Z(P)$ is nontrivial.

3. Let $R$ be a commutative ring with identity and let $M$ be a maximal ideal of $R$.
   (a) Show $M[x]$ is an ideal of $R[x]$ (for $x$ an indeterminate).
   (b) Show $M[x]$ is a prime ideal, but not a maximal ideal.
   (c) Find a maximal ideal of $R[x]$ that contains $M[x]$.

4. Let $p(x) = x^5 - 2$, $K$ the splitting field (in $\mathbb{C}$) of $p(x)$ over $\mathbb{Q}$, and let $G = Gal(K/\mathbb{Q})$.
   (a) Find the order of $G$ and a set of generators. Is $G$ abelian?
   (b) Find the intermediate field $\mathbb{Q} \subset E \subset K$ such that $E$ is not a normal extension of $\mathbb{Q}$. Prove this in two ways: using the definition and the Fundamental Theorem of Galois Theory.
   (c) Find an extension $F$ of $\mathbb{Q}$ of degree two such that $\mathbb{Q} \subset F \subset K$, and a set of generators of $Gal(F/K)$.

5. Let $V$ and $W$ be modules over a ring $F$, and let $T \in \text{Hom}_F(V,W)$. Let $Z = \{(v, T(v)) | v \in V \}$.
   (a) Show that $V \times W$ is a module over $F$.
   (b) Show that $Z$ is a submodule of $V \times W$.
   (c) If $F$ is a field, $V$ is a vector space of dimension $n$, $W$ is a vector space of dimension $m$ and $T$ is of rank $r$, what is the dimension of $Z$? Verify your claim.

6. Let $R$ be a commutative ring with 1. Recall that $R$ is local if $R$ has a unique maximal ideal.
   (a) Show that $R$ is local if and only if the set of all nonunits of $R$ is an ideal in $R$.
   (b) Suppose $R$ is local, $S$ is a nonzero commutative ring with 1, and $f : R \to S$ is a surjective ring homomorphism. Show that $S$ is a local ring.

7. Let $G$ be a group, with center $Z(G)$.
   (a) Suppose $G/Z(G)$ is cyclic. Prove that $G$ is abelian.
   (b) Suppose $G$ has order 441. Prove that $G$ is solvable. (Prove every claim/result you utilize in your solution, except for named theorems.)