Instructions: Do all nine problems. You will have six hours for this exam.

1. Let $P$ be a $p$-Sylow subgroup of a group $G$ and let $H$ be a subgroup of $G$ containing $P$.
   (a) Show that if $P 	riangleleft H$ and $H 	riangleleft G$, then $P 	riangleleft G$.
   (b) Show that $N_G(N_G(P)) = N_G(P)$.

2. Let $R$ be a commutative ring with 1. If $M$ is an $R$-module, let $\text{End}_R(M)$ denote the set of $R$-module homomorphisms from $M$ into $M$.
   (a) Show that $\text{End}_R(M)$ is a ring (not necessarily commutative) with 1. (You must define the operations on $\text{End}_R(M)$.)
   (b) Show that $\text{End}_R(R) \cong R$ as rings.
   (c) Generalize (b) to $\text{End}_R(R/I)$, where $I$ is an ideal in $R$.

3. Let $R$ be a commutative ring and $I$ an ideal in $R$. The radical $\sqrt{I}$ of $I$ is defined by
   \[ \sqrt{I} = \{ r \in R \mid r^m \in I \text{ for some integer } m \geq 1 \}. \]
   An ideal $J$ of $R$ is called a radical ideal (or just radical) if $J = \sqrt{J}$.
   (a) Prove that $\sqrt{I}$ is an ideal in $R$.
   (b) Prove that every prime ideal in $R$ is radical.
   (c) Prove that $\sqrt{J}$ is radical.
   (d) Suppose $R$ is not the zero ring. Prove that $I$ is radical if and only if $R/I$ has no nilpotent elements.

4. Let $K$ be the splitting field of $x^6 - 25$ over $\mathbb{Q}$. Determine $\text{Gal}(K/\mathbb{Q})$. Explicitly determine all subfields of $K$, giving generators over $\mathbb{Q}$. Indicate which are Galois over $\mathbb{Q}$.

5. Let $K$ be an extension field of a field $F$, and suppose $\alpha \in K$ is algebraic over $F$. Prove that $F(\alpha) \cong F[x]/(f(x))$, where $f(x)$ is an irreducible, monic polynomial of degree $n \geq 1$ satisfying $f(\alpha) = 0$. (NOTE: You may not presume the existence of such an $f(x) \in F[x]$.)

6. Let $R$ and $S$ be commutative rings with identity, and let $M$ be a module for $R$.
   (a) State and prove a condition on $M$ in order for $M$ to be cyclic. (Recall that a module is cyclic if it is generated by a single element.)
   (b) Let $\psi : R \to S$ be a surjective ring homomorphism. Prove that $S$ is a cyclic $R$-module.
   (c) Now give an example of a ring $R$ and an $R$-module $M$ for which $M$ is not cyclic.

7. For four subgroups $A, B, A', B'$ of a group $G$, let $A' \triangleleft A$ and $B' \triangleleft B$.
   (a) Show that $A' \cap B \triangleleft A \cap B$, and that $A' \triangleleft (A \cap B)A'$.
   (b) Show that $(A' \cap B)(A \cap B') \triangleleft (A \cap B)$.
   (c) Show that $(A \cap B')A' \triangleleft (A \cap B)A'$. (HINT: Define a homomorphism \[ \psi : A \cap B \to (A \cap B)A'/A', \] and then consider $\psi((A \cap B')(A' \cap B))$.)

Algebra Preliminary Exam
December 17, 2003