ALGEBRA PRELIM
WINTER 2000

1. a. Show that a group $G$ of order 56 is not simple.
   b. Show that a group $G$ of order 72 is not simple.

2. Prove Cauchy’s Theorem: If $G$ is a finite group and $p$ a prime divisor of $G$, then $G$ contains an element of order $p$.
   \textit{Hint:} First do the Abelian case, then use the class equation.

3. Let $G$ be a group. Let $Z(G)$ be its center, $Aut(G)$ its automorphism group, and $Inn(G)$ be its inner automorphism group. An inner automorphism is one that is induced by conjugation by an element of $G$
   a. Show $Inn(G)$ is a normal subgroup of $Aut(G)$.
   b. Show $G/Z(G) \cong Inn(G)$.
   c. Show that if $Inn(G)$ is cyclic, then $G$ is Abelian.

4. Let $G$ be a group and $H$ a subgroup of finite index. Show that there exists a normal subgroup $N$ of $G$ contained in $H$ and also of finite index.
   \textit{Hint:} If $[G : H] = n$, find a homomorphism of $G$ into $S_n$ whose kernel is contained in $H$.

5. a. Let $F$ be a field and $a, b \in F$. Show that if $a^m = b^m$ and $a^n = b^n$, for $m$ and $n$ relatively prime positive integers, then $a = b$.
   b. Prove the same statement when the field $F$ is replaced by a (commutative) integral domain $D$.

6. Let $F$ be a field and $f(x) \in F[x]$ be a monic polynomial of positive degree.
   a. Show that there exists an extension field $E$ of $F$ ($F \subseteq E$) such that $E$ contains a root of $f(x)$.
   b. Show that $f(x)$ has a splitting field $K$ over $F$.

7. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. That is, $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z} \text{ and } i^2 + 1 = 0\}$.
   In the ring $\mathbb{Z}[i]$ let $A$ be the principal ideal generated by $1 + 3i$. Prove that $\mathbb{Z}[i]/A$ is isomorphic to $\mathbb{Z}_{10}$, the ring of integers modulo 10.

8. Let $K$ be the splitting field of $x^4 + 1$ over $\mathbb{Q}$. Show that $K$ is a simple extension $K = \mathbb{Q}(\alpha)$, and find the Galois group of $K$ over $\mathbb{Q}$. (You should be able to identify the Galois group up to isomorphism with a well-known group.)

9. Consider the quotient ring $F[x]/(x^2 + 1)$ for each of the following fields $F = \mathbb{Z}_2$, $F = \mathbb{Z}_3$, $F = \mathbb{Q}$, and $F = \mathbb{C}$, and determine in which of the four cases the quotient ring is a field.
10. For each statement indicate TRUE or FALSE with brief justification:
   a. If $G$ is the group of invertible $2 \times 2$-matrices with entries in $\mathbb{F}_q$ the field of $q$ elements, then $G$ has order $(q^2 - 1)(q^2 - q)$.
   b. If $H$ and $K$ are normal subgroups of a group $G$ and $G/H \cong G/K$, then $H \cong K$.
   c. If $p$ is prime, then the group of units in the ring $\mathbb{Z}/p^n\mathbb{Z}$ of integers modulo $p^n$ is cyclic.
   d. An irreducible polynomial $f(x)$ over a field $K$ has no repeated roots in any extension field of $K$.
   e. Every unique factorization domain is a principal ideal domain.