**Instructions:** There are seven questions, some with several parts. Write your solution to each problem on a separate sheet of paper, with your name at the top.

(1) For each \( n \), let \( S_n \) be the symmetric group on \( n \) letters, and let \( G \) be a finite group. Prove the following:
   (a) If the order \(|G|\) of \( G \) is a product of two distinct primes then \( G \) is not simple.
   (b) If \(|G| = p^2q\) for \( p \) and \( q \) distinct primes, then \( G \) is not simple.
   (c) If \( n \leq 4 \), then \( S_n \) has no non-abelian simple subgroups.
   (d) If \( G \) is a non-abelian simple group and \( H \) is a proper subgroup of \( G \), then \(|G : H| \geq 5\).

(2) Let \( V = \mathbb{R}^4 \). Let \( S \) be the set of all two-dimensional subspaces of \( V \), and fix \( W \in S \). Let \( G = GL(V) \) (the group of invertible linear operators on \( V \)) act naturally on \( S \), and let \( H = \{ g \in G : g \cdot W = W \} \). Show that \( H \) has exactly three orbits on \( S \).

(3) Let \( k \) be a field, and let \( R \) be the ring of \( n \times n \) matrices with entries in \( k \). Let \( V = k^n \). Show that \( V \) is a simple \( R \)-module, with each element of \( R \) acting by the associated linear transformation.

(4) Let \( R \) be a commutative ring with identity, \( I \) an ideal of \( R \), and \( M \) a module over \( R \). Let \[
S = \{ a \cdot m : a \in I, m \in M \}.
\]
   (a) Is \( S \) an \( R \)-submodule of \( M \)? Prove or give a counterexample.
   (b) Let \( I \cdot M \) be the \( R \)-submodule of \( M \) generated by \( S \). Prove that \( M \otimes_R (R/I) \cong M/(I \cdot M) \).

(5) Let \( R \) be commutative ring with identity. The **Krull dimension** of \( R \), denoted by \( \text{dim}(R) \), is the supremum of the indices \( d \) of all strictly increasing chains
   \[
P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d
\]
   of prime ideals in \( R \). Define \( \text{dim}(R) = \infty \) if there is no finite supremum.
   (a) Determine the Krull dimension \( \text{dim}(R) \) if \( R \) is a field.
   (b) Determine the Krull dimension \( \text{dim}(R) \) if \( R = \mathbb{Z} \).
   (c) Show that an ideal \( P \) in \( R \) is prime if and only if \( R/P \) is an integral domain.
   (d) Show that \( \text{dim}(R) \geq n \) if \( R = k[x_1, \ldots, x_n] \) for \( k \) a field.

(6) Let \( E = \mathbb{Q}(a) \) for \( a = \sqrt{1 + \sqrt{2}} \).
   (a) Find \( |E : \mathbb{Q}| \).
   (b) Identify \( \text{Gal}(E/\mathbb{Q}) \).
   (c) How many subfields of \( E \) are there?

(7) Consider the polynomial \( f(x) = x^7 - 1 \in \mathbb{F}_2[x] \).
   (a) Find a splitting field \( K \) for \( f(x) \).
   (b) Factor \( f(x) \) into irreducible polynomials over \( \mathbb{F}_2[x] \).
   (c) Show that the squaring map \( \varphi(a) = a^2 \) is an automorphism of \( K \) fixing \( \mathbb{F}_2 \), and find the orbits of \( \varphi \).