Instructions. Work all of these problems and write their solutions clearly and completely (and legibly). Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. You have 6 hours to complete this exam.

1. Let \( R \) be a commutative ring with 1, and let \( I \) and \( J \) be ideals in \( R \) with \( I + J = R \).
   
   (a) Let \( a, b \in R \). Prove that there exists \( c \in R \) such that \( c \equiv a \mod I \) and \( c \equiv b \mod J \).
   
   (b) Deduce from the above that \( R/(I \cap J) \) is isomorphic to the direct product \( R/I \times R/J \).

2. Show that any linear operator on a finite dimensional vector space (over a field of characteristic not equal 2) which satisfies \( T^2 = I \) is diagonalizable.

3. Let \( G \) be a group, and let \( N \) be a normal subgroup of \( G \). Let \( \phi : G \to G/N \) be the canonical homomorphism. Let \( H \) be another group, and \( f : G \to H \) be a homomorphism.
   
   (a) Show that there is a homomorphism \( \bar{f} : G/N \to H \) such that \( \bar{f} \circ \phi = f \) if and only if \( N \leq \text{Ker}(f) \).
   
   (b) Assuming \( \bar{f} \) exists, show that it is unique.

4. Consider the polynomial \( f = x^8 - 1 \in \mathbb{F}_3[x] \).
   
   (a) Find a splitting field \( K \) for \( f \).
   
   (b) Factor \( f \) into irreducible polynomials over \( \mathbb{F}_3[x] \).
   
   (c) Show that the cubing map \( \phi(a) = a^3 \) is an automorphism of \( K \) fixing \( \mathbb{F}_3 \), and find the orbits of the group generated by \( \phi \).

5. Let \( G \) be a group of order \( p^2 q \), where \( p \) and \( q \) are distinct primes. Show that \( G \) is not simple.

6. Suppose \( R \) is a ring with 1 and \( M \) is a left \( R \)-module. Let \( N_1 \subseteq N_2 \subseteq \cdots \) be an ascending chain of submodules of \( M \). Show that \( \bigcup_{i=1}^{\infty} N_i \) is a submodule of \( M \).