Math Prize Competition!
Western Michigan University
October 26, 2019

Name (printed legibly): _____________________________________________
Wmich email address: ____________________________________________

Read all of the following before working on the contest problems:

• This contest is **closed book**. Discussing the problems with colleagues is **not** permitted during the contest. The use of a calculator is **not** allowed for this contest. The use of a cell phone is **not** allowed for this contest. A violation of these rules results in immediate disqualification.

• Show or explain all of your work. Please write clearly and completely. Unjustified answers are regarded as incorrect.

• Please keep your written answers brief; be clear and to the point. (Do not ramble.)

• The problems are not necessarily in order of difficulty. If you spend more than 5–10 minutes thinking about a problem with no progress, you should probably look at the next one to see if it’s easier.

• This contest has 6 problems and each problem is worth 10 points. If you do not have every page, please inform a proctor.

• You have three hours to complete your work on these problems.

• If you need clarification on any problem, please ask a proctor.

• Good luck!

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Problem 1. Suppose that Dr. Ziebarth has four keys and that each key unlocks exactly one door. If he inserts the keys at random, one in each door, what is the probability that he will unlock exactly two doors?

Problem 2. Let

\[ a_0 = \frac{1 + \sqrt{5}}{2}, \quad a_{n+1} = a_n^2 - 2 \quad \text{for } n \geq 0. \]

Find \( a_{2019} \).

Problem 3. Suppose \( S \) is a set of irrational real numbers such that the sum of any two different numbers in \( S \) is rational. Show that there are at most two numbers in \( S \).

Problem 4. Find all possible pairs of positive integers \( (m, n) \) such that

\[ 1 + 5 \cdot 2^m = n^2. \]
Problem 5. \( \triangle ABC \) is an equilateral triangle, and \( P \) is a point inside the triangle. We draw three line segments through \( P \): one of length \( x \) perpendicular to \( AB \), one of length \( y \) perpendicular to \( BC \) and one of length \( z \) perpendicular to \( AC \).

\[ \begin{align*}
  & \text{a) Prove that } x + y + z \text{ has the same value regardless of where the point } \\
  & \hspace{1em} \text{for } P \text{ is.} \\
  & \hspace{1em} \text{Hint: Think about areas.} \\
  & \text{b) Suppose } \triangle ABC \text{ has area } a. \text{ What is the area of the region containing} \\
  & \hspace{1em} \text{all points } P \text{ such that it is possible to form a triangle with sides of} \\
  & \hspace{1em} \text{length } x, y \text{ and } z? \text{ In other words we mean the points } P \text{ such that} \\
  & \hspace{1em} x + y \leq z, \quad x + z \leq y, \quad y + z \leq x.
\end{align*} \]

Problem 6. There are \( n \) people at a party, whose names are \( p_1, p_2, \ldots, p_n \). For any two people \( p_i, p_j \) there are two possibilities: either they know each other or they do not (and if \( p_i \) knows \( p_j \) we assume that \( p_j \) also knows \( p_i \)). The people do not know themselves. For each \( i \), let \( k_i \) be the number of people at the party who know the person \( p_i \).

\[ \begin{align*}
  & \text{a) Suppose } n = 4 \text{ and } k_i = i \text{ for } i = 1, 2, 3. \text{ Prove that } k_4 = 2. \\
  & \text{b) Suppose } n = 4\ell \text{ for some integer } \ell \geq 1 \text{ and } k_i = i \text{ for all} \\
  & \hspace{1em} 1 \leq i \leq n - 1. \text{ Prove that } k_n = \frac{n}{2}.
\end{align*} \]