Analysis Prelim

August 25, 2014

Solve any 5 of the next 7 problems

1. (a) State Lusin's Theorem.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such $f = g$ a.e. (with respect to the Lebesgue measure).

2. (a) Give the definition of weak convergence in a Banach space.

   (b) Give an example of a (strongly) closed set that is not weakly closed and justify your example.

3. Let f be an absolutely continuous function on $[0, 1]$, such that $f(0) = 0$ and

$$\int_0^1 |f'(x)|^2 dx < \infty.$$

Prove that

$$\lim_{x \to 0+} \frac{f(x)}{\sqrt{x}}$$

exists and calculate the value of this limit.

4. Calculate

$$\lim_{n \to \infty} \int_0^1 n^3 x^{3/4} (1 + n^4 x^2)^{-1} \, dx,$$

and justify your work.

5. (a) Give the definition of the convergence in measure.

   (b) Let $\{f_n\}$ be a sequence of functions on $\mathbb{R}$ that converges to a function f in measure, let $p \geq 1$, let $g \in L^p(\mathbb{R})$ and suppose that $|f_n(x)| \leq g(x)$, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Prove that $\{f_n\}$ converges to f in the norm of $L^p(\mathbb{R})$.

   (c) Give an example of a a sequence $\{f_n\}$ in $L^p(\mathbb{R})$ that converges to a function f in measure but not in the norm of $L^p(\mathbb{R})$.

6. (a) State Fubini's Theorem.

   (b) Apply Fubini's Theorem to calculate

$$\int_E \frac{y}{x} e^{-x} \sin x \, d\mu$$

   where μ is the product of Lebesgue measure on $\mathbb{R}$ with itself, and $E = \{(x, y) : 0 \leq y \leq \sqrt{x}\}$.

7. Let $\{f_n\}$ and f be functions in $L^2(\mathbb{R})$. Prove that $\{f_n\}$ converges to f strongly (in the norm of $L^2(\mathbb{R})$) if and only if $\{f_n\}$ converges weakly to f and $\|f_n\| \to \|f\|$.