Analysis Prelim

March 13, 2015

SOLVE ANY 5 OF THE NEXT 8 PROBLEMS

1. Suppose that f is a function of bounded variation on $[0, 1]$, and let $V(x)$ be the total variation function for f, i.e., for any $x \in [0, 1]$, $V(x)$ is the total variation of f on the interval $[0, x]$. Prove that, if V is absolutely continuous on $[0, 1]$, then so is f.

2. Let (X, β, μ) be a finite measure space and let $f \in L^1(\mu) \cap L\infty(\mu)$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

3. (a) State the Riesz Representation Theorem for the Dual of $L^p(E)$.

   (b) Let E be a measurable set, let p and q be positive numbers such that $1/p + 1/q = 1$, and let S be a dense subset of $L^q(E)$. Show that if $g \in L^p(E)$ and $\int_E fg = 0$ for all $f \in S$, then $g = 0$.

4. Suppose that λ and μ are σ-finite measures on the same σ-algebra $\mathcal{M}$, and that λ is absolutely continuous with respect to μ. Define ν on $\mathcal{M}$ by $\nu(E) = \lambda(B \cap E)$ where B is a fixed member of $\mathcal{M}$.

   (a) Prove that ν is a measure on $\mathcal{M}$.

   (b) Prove that ν is absolutely continuous with respect to μ.

   (c) Find the Radon-Nikodym derivative of ν with respect to μ.

5. (a) State the Uniform Boundedness Principle.

   (b) Prove that every weakly convergent sequence in Hilbert space must be bounded.

6. Let (X, μ) be a σ-finite complete measure space and let $f : X \to [0, \infty)$ be measurable. Prove that

$$\int_X f \, d\mu = \int_0^\infty \mu\left(\{x \in X : f(x) \geq t\}\right) \, dt.$$

7. (a) Give the definition of a measurable space (X, β).

   (b) Define what it means for a real valued function f to be measurable on a measurable space (X, β).

   (c) Suppose that f is measurable on a measurable space (X, β) and B is a Borel set in the real line. Prove or disprove: $f^{-1}(B) \in \beta$.

8. Find a sequence of real valued nonnegative function f_n on $[0, 1]$ so that

$$\limsup_{n \to +\infty} f_n(x) = +\infty, \quad \forall x \in [0, 1]$$

and

$$\lim_{n \to +\infty} \int_{[0,1]} f_n(x) dx = 0.$$