Analysis Prelim

August 30, 2016

Solve any 5 of the next 7 problems

1. Define $E = \{x \in [0,1] : \left|x - \frac{p}{q}\right| < q^{-3}$ for infinitely many $p, q \in \mathbb{N}\}$. Prove that $m(E) = 0$.

2. Let $f : [0,1] \to \mathbb{R}$, and suppose that the set $\{x \in [0,1] : f(x) = c\}$ is (Lebesgue) measurable for every $c \in \mathbb{R}$. Prove or disprove: f is a (Lebesgue) measurable function.

3. Let f be a function defined on $[0,1]$ that is integrable over $[0,1]$, differentiable at $x = 0$, and $f(0) = 0$. Let

$$g(x) = \begin{cases} x^{-3/2} f(x), & \text{if } 0 < x \le 1, \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that g is integrable over $[0,1]$.

4. Calculate the Lebesgue integral

$$\int_0^2 \int_{y^2}^4 y \cos(x^2)\, dx\, dy.$$

5. Let $\{x_n\}$ be an unbounded sequence in Hilbert space $\mathcal{H}$. Prove that there exists a vector $x \in \mathcal{H}$ such that the sequence $\{\langle x_n, x\rangle\}$ is unbounded.

6. Calculate

$$\lim_{n\to\infty} \int_0^\infty \frac{1}{1 + x^{\frac{\sqrt{n}}{\ln(n+2016)}}}\, dx,$$

and justify your work.

7. Let X be the set of functions of bounded variations on $[0,1]$, with the property that $f(0) = 0$. Let $TV(f)$ denote the total variation of $f \in X$ over the interval $[0,1]$.

(a) Prove that $TV(f)$ is a norm on X.

(b) Prove that X is complete with respect to this norm.