Analysis Prelim

February 11, 2017

Solve any 5 of the next 8 problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that f is continuous. Let A be a Lebesgue measurable subset of $\mathbb{R}$. Prove or disprove: $f^{-1}(A)$ is a Lebesgue measurable set.

2. Let $\{f_n\}$ be a sequence of integrable functions on $[0,1]$, let f be an integrable function on $[0,1]$, and suppose that $f_n \to f$ a.e. on $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} |f - f_n| = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \int_{[0,1]} |f_n| = \int_{[0,1]} |f|.$$

3. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$, and suppose that $f_n \to f$ in measure on $[0,1]$. Prove that there exists a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on $[0,1]$ to f.

4. Let f be an integrable function on $\mathbb{R}$. For $t \in \mathbb{R}$, define

$$F(t) = \int_{\mathbb{R}} \sin(tx) f(x)\, dx.$$

(a) Show that F is continuous on $\mathbb{R}$.
(b) Prove that $\lim_{t\to+\infty} F(t) = 0$.

5. Let f and g be integrable functions on $\mathbb{R}$ with finite supports. Let

$$h(t) = \int_{\mathbb{R}} f(s)g(t-s)ds, \quad \text{and} \quad p(t) = t,$$

for all $t \in \mathbb{R}$. (You may assume that h is well-defined.) Prove the implication

$$\int_{\mathbb{R}} pf = \int_{\mathbb{R}} pg = 0 \quad \Longrightarrow \quad \int_{\mathbb{R}} ph = 0.$$

6. Give an example of a sequence of real valued nonnegative functions $\{f_n\}$ on $[0,1]$ so that
(a) $\limsup_{n\to+\infty} f_n(x) = +\infty$, for all $x \in [0,1]$,
(b) $\lim_{n\to+\infty} \int_{[0,1]} f_n = 0$.
Prove that your example satisfies both conditions.

7. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. If a sequence $\{x_n\} \subset \mathcal{X}$ is weakly convergent, prove that the same is true for $\{Tx_n\}$.

8. Let $\{f_n\}$ be a sequence of functions on $[0, 1]$ such that:

(a) f_n is absolutely continuous, for each $n \in \mathbb{N}$;

(b) $f_n(0) = 0$, for each $n \in \mathbb{N}$;

(c) the sequence $\{f'_n\}$ converges weakly in $L^1([0, 1])$.

Prove that:

(A) the sequence $\{f_n(x)\}$ is convergent for all $x \in [0, 1]$;

(B) the limit of $\{f_n\}$ is absolutely continuous.

Give an example to demonstrate that the result does not hold if *absolutely continuous* is replaced by *bounded variation* both in the assumption (a) and in the conclusion (B).