# Analysis Prelim

August 27, 2018

**Solve any 5 of the following 7 problems**.

1. Let $A$ be the set of irrational numbers in the interval $[0,1]$. Prove that $m^*(A) = 1$.

2. Let $A := [a, b]$. Suppose that the $f : A \to \mathbb{R}$ is continuous, $g : A \to \mathbb{R}$ is integrable and $g(x) \geq 0$ for almost all $x \in A$.

(a) Show that the function $f(x)g(x)$ is integrable.

(b) There exists a point $p \in A$ such

$$\int_A f(x)g(x)\ dx = f(p) \int_A g(x)\ dx \tag{1}$$

(c) Is (1) valid in the case $A = [a, b] \cup [c, d]$ if $[a, b] \cap [c, d]] = \emptyset$.

3. Let $f$ be a function defined on $[0, 1]$ in the following way. If $x$ belongs to the Cantor set, then $f(x) = 0$. If $x$ belongs to a complementary interval of length $3^{-k}$, then $f(x) = k$. Find $\int_{[0,1]} f$.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative functions in $L^2(0, 1)$, and suppose that $\{f_n\}$ converges to a function $f$ in the norm of $L^2(0, 1)$. Prove that $f \geq 0$. Does the statement remain true if $\{f_n\}$ converges *weakly* to $f$?

5. Recall that $\ell_2 := \{x = (x^1, x^2, \dots) : \sum_{k=1}^{\infty} x_k^2 < +\infty\}$ with norm $\|x\| := \sqrt{\sum_{k=1}^{\infty} x_k^2}$ is a Hilbert space. We consider the following ellipse

$$E_a = \{x = (x^1, x^2, \dots) \in \ell_2 : \sum_{k=1}^{\infty} \frac{(x^k)^2}{a_k^2} \leq 1\}$$

(a) Show that the ellipse $E_a$ is not sequentially compact for the case

$$a_k = 1, \quad k = 1, 2, \dots$$

(b) Show that the ellipse $E_a$ is sequentially compact for the case

$$\sum_{k=1}^{\infty} a_k^2 < +\infty.$$

6. Let $f_n$ be a sequence of nonnegative measurable functions on $[0,1]$. Moreover, suppose that $\lim_{n\to\infty} \int_0^1 f_n(x) d\mu = 0$.

(a) Prove or disprove: $f_n$ converges to 0 in measure on $[0,1]$; and

(b) Prove or disprove: $f_n$ converges to 0 almost everywhere on $[0,1]$.

7. Calculate

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{y}{x}\right) dx dy.$$