Analysis Preliminary Examination
May 2019

Name:

Select ONLY 5 out of the following 7 problems:

1. Let f be a differentiable function such that $|f'(x)| \leq k$ for all $x \in R$ where k is a fixed constant. Define a sequence $\{x_n\}$ recursively by $x_1 = f(1)$ and $x_{n+1} = f(x_n)$. Prove or disprove: the sequence $\{x_n\}$ converges for the following two cases: (a) $k < 1$ and (b) $k = 1$.

2.

(a) Prove that if $\sum_{n=1}^{\infty} a_n^2$
converges then so does $\sum_{n=1}^{\infty} a_n/n$.

(b) Does the converse of (a) hold? Justify your answer.

3. (a) State the definition of a function to be measurable. (b) Prove that if both f and g are measurable functions then so is $f \cdot g$.

4. (a) Verify that $e^{-xy} \sin x$ is an integrable function on $[0, \infty) \times [0, \infty)$.
(b) Show that $\int_0^{\infty} \sin x/x dx = \pi/2$. (Hint: Apply Fubini's thepreem to the function in part (a)).

5. (a) State the definition of linear bounded operator $A : X \to Y$ for Banach spaces X and Y.
(b) Is the following operator $A : C[0, 1] \to C[0, 1]$

$$Ax(t) := \int_0^t x(s)\, ds$$

linear and bounded? If it is bounded find its norm (note that $C[0, 1]$ is a space of continuous functions $x(t)$ defined on the interval $[0, 1]$ with the norm $\|x\| := \max_{t \in [0,1]} |x(t)|$).

(c) Is the following operator $A : L_1[0,1] \to L_1[0,1]$

$$Ax(t) := x(\sqrt{t})$$

linear and bounded? If it is bounded find its norm (note that $L_1[0,1]$ is a space of integrable functions $x(t)$ defined on the interval $[0,1]$ with the norm $\|x\| := \int_{[0,1]} |x(t)|\ dt$).

6. Let L be a finite-dimensional subspace of normed linear vector space X. This means that there are vectors $e_1, e_2, \ldots, e_n$ in L such that any $z \in L$ is represented as a linear combination

$$z = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$$

Show that for any $x \in X$ there exists vector $y \in L$ such that

$$\|x - y\| = \inf_{z \in L} \|x - z\|$$

7. Let a sequence of measurable functions $f_n(t)$ on the interval $[0,1]$ converge to a function $f(t)$ for almost all $t \in [0,1]$ and for some constant M we have that for all n and almost all $t \in [0,1]$

$$|f_n(t)| \leq M$$

Show that the sequence of functions

$$\phi_n(x) := \int_0^x (f_n(t) - f(t))\ dt$$

converges uniformly to zero on the interval $[0,1]$