# Analysis Preliminary Examination
# October 24, 2020

**Solve any 5 of the following 8 problems.**

1. Prove or disprove: If $A, B$ are Lebesgue measurable subsets of $\mathbb{R}^2$ then so is $A+B$. Here $A+B = \{x+y : x \in A, y \in B\}$.

2. Prove or disprove: The set of continuous function on $[0,1]$ is dense in $L^\infty[0,1]$.

3. Let $X$ be the normed linear space whose elements are all continuous function on $[0,1]$, and the norm is given by

$$\|f\| = \int_0^1 |f|.$$

Prove that $X$ is not a Banach space.

4. Let $\{f_n\}$ be a sequence in $L^2[0,2\pi]$ defined by $f_n(x) = \sin nx$. Prove that $\{f_n\}$ converges weakly but not strongly (i.e., in the norm of $L^2[0,2\pi]$).

5. Suppose that $\mu$ and $\nu$ are finite measures on a measureable space $(X, \mathfrak{M})$. Prove that there exists a nonnegative measurable function $f$ on $X$ such that for all $E \in \mathfrak{M}$

$$\int_E (1-f)\,d\mu = \int_E f\,d\nu.$$

6. Let $m^*$ be the Lebesgue outer measure on $\mathbb{R}$. Prove that a bounded set $E \subset \mathbb{R}$ is Lebesgue measurable if and only if for any $\varepsilon > 0$ there exists a closed subset $F \subset E$ such that $m^*(E-F) < \varepsilon$.

7. Let $A$ be a subset of $\mathbb{R}$ with finite Lebesgue measure, and let $f$ be a measurable function on $A$. Prove that $f$ is integrable on $A$ if and only if both of the following series converge.

$$\sum_{k=1}^{\infty} k\, m(\{x \in A \ : k \le |f(x)| < k+1\})$$

$$\sum_{k=1}^{\infty} m(\{x \ : |f(x)| \ge k\})$$

8. Let $f$ and $g$ be two measurable functions on a measurable space $(X, \mathfrak{M})$. Prove that their product $fg$ is also measurable.