Preliminary Exam in Algebra
Department of Mathematics, Western Michigan University
24 August 2021

Instructions. Work all of the following six problems. Write your solutions on the paper provided. Start each solution at the top of a new sheet of paper, indicating the number of the problem on that page, and write your name on each page. Solutions should be clear and complete. You have 6 hours to complete this exam. No aids such as books, notes or internet access are permitted.

1. Let $R \subset B$ be integral domains and assume that B is a free R-module of finite rank $n \geq 2$.
   (a) Show that B is isomorphic to a subalgebra of the R-algebra $M_n(R)$, the ring of $n \times n$ matrices over R. Hint: Given $y \in B$, consider the function $\mu_y : B \to B$ defined by $\mu_y(x) = yx$.
   (b) Define the corresponding norm function $\nu : B \to R$ and show that it is independent of the R-basis of B.
   (c) Let $p \in \mathbb{Z}$ be a prime integer. By explicit construction, find a subring of $M_2(\mathbb{Z})$ that is isomorphic to $B = \mathbb{Z}[\sqrt{p}]$.
2. Determine with justification all groups of order 21 up to isomorphism.
3. Fix the integer $n \geq 1$ and let S_n be the symmetric group on n letters. Define the field

   $$E = \mathbb{Q}(x_1, \ldots, x_n) \cong \mathbb{Q}^{(n)}$$

   where $\mathbb{Q}$ is the field of rational numbers and let S_n act on E by fixing $\mathbb{Q}$ and:

   $$\sigma(x_i) = x_{\sigma(i)} \quad , \ \sigma \in S_n \ , \ 1 \leq i \leq n$$

   (a) Define the **elementary symmetric polynomials** $e_1, \ldots, e_n \in \mathbb{Q}[x_1, \ldots, x_n]$ where $\deg e_i = i$. Hint: Given $f \in E$ consider the polynomial $\frac{1}{n!} \sum_{\sigma \in S_n} \sigma(f)$.
   (b) Give a set of generators over $\mathbb{Q}$ for the fixed field E^{S_n}. (You do not need to prove your answer.)
   (c) Let $F = \mathbb{Q}(e_1, \ldots, e_n)$ and let $\bar{F}$ be the algebraic closure of F. Show that $F \subset E \subset \bar{F}$.
   Hint: Consider the polynomial $f(T) = (T - x_1) \cdots (T - x_n) \in E[T]$.
4. Let F be a field and F^* the group of units of F.
   (a) Show that every finite subgroup G of F^* is cyclic.
   Hint: Use the Fundamental Theorem for Finitely Generated Abelian Groups to show that, if G is not cyclic, then there exists $d < |G|$ with $x^d = 1$ for all $x \in G$.
   (b) Prove or give a counterexample: Every finitely generated subgroup of F^* is cyclic.
5. Show that a unique factorization domain (UFD) is a normal domain, i.e., integrally closed in its field of fractions.
6. Let G be a group and H a subgroup. Recall the following definitions.
   1. The **normalizer** of H in G is $N_G(H) = \{x \in G \,|\, xHx^{-1} = H\}$. Equivalently, $N_G(H)$ is the stabilizer of H for G acting on subgroups by conjugation.
   2. The **normal hull** of H in G is the subgroup H^G generated by the set $\{xhx^{-1} \,|\, x \in G \,,\, h \in H\}$.

   (a) Show that $N_G(H)$ is the largest subgroup S of G such that H is a normal subgroup of S.
   (b) Show that H^G is the smallest normal subgroup of G containing H.