WMU Department of Mathematics
Algebra Comprehensive Exam
August 25, 2017
(Version 8/18/17)

Instructions. Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. Please write clearly and legibly. You have 6 hours to complete this exam.

1. Let p be a prime, and let G be a group of order p^4. Assume that $Z(G)$ has order p^2. Find the number of conjugacy classes in G.

2. Let Ω be a set, G a group acting on Ω, and $\Omega^{(2)}$ the set of all pairs of distinct elements of Ω. We say that G acts *sharply 2-transitively* on Ω if for every pair of elements $a = (x, y)$ and $b = (z, w)$ of $\Omega^{(2)}$, there is a *unique* element of G taking a to b.

   Let F be a field, and let

   $$G = \{f : F \to F : f(x) = mx + b \text{ for some } m \neq 0, b \in F\}.$$

   (a) Show that G is a group that acts sharply 2-transitively on F.

   (b) Exhibit G as a semi-direct product of $(F, +)$ and F^*.

3. Suppose R is a Unique Factorization Domain (UFD), let K be the field of fractions of R, and let $f \in R\backslash\{0\}$. Show that the subset

   $$R_f = \left\{\frac{r}{f^n} : r \in R, n \in \mathbb{Z}\right\} \subset K$$

   is a UFD.

4. Let $\mathbb{F}_3$ denote the field with three elements. Are the rings $\mathbb{F}_3[x]/(x^3 - x - 1)$ and $\mathbb{F}_3[x]/(x^3 + x^2 - 1)$ fields? Are they isomorphic? If not, why not? If yes, give an explicit isomorphism between them.

5. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} T = 0$ for every torsion abelian group T. Recall that an abelian group is a *t*orsion group if all elements have finite order.

6. Let $\zeta = e^{\frac{2\pi i}{37}}$, and $\alpha = \zeta + \zeta^{10} + \zeta^{26}$. Determine the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

7. Let A and B be 3×3 matrices over a field F. If A and B have the same minimal polynomial and the same characteristic polynomial, prove that they are similar. Also, give an example of two 4×4 matrices over some field F with the same minimal and characteristic polynomials which are not similar.

8. Prove that every vector space has a basis. (This requires Zorn's lemma.)