# Graph Theory Preliminary Examination

## January 6, 2018

### Instructions

Do **exactly four** of the six problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which problem in Part A and which two problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

## Part A

A1 The crossing number of a graph $G$ is the smallest possible number $cr(G)$ of edge crossings in any drawing of $G$ in the plane (so, for example, $G$ is planar iff $cr(G) = 0$). Show that $cr(K_{3,4}) = 2$.

*Hint:* To show that $cr(K_{3,4}) \neq 1$, suppose $K_{3,4}$ is drawn in the plane with only one pair of edges crossing. Form a new graph $G$ by putting a vertex at the point where these edges intersect.

A2 Suppose $G$ is a 3-regular graph whose edge-chromatic number is $\chi'(G) = 3$. Suppose further that the partition of $E(G)$ into three matchings is unique. Prove that $G$ has a Hamilton cycle.

A3 a) Suppose $G$ is 2-connected and $u, v$ are distinct vertices in $G$. Show that there is a cycle containing both $u$ and $v$.

b) Suppose $G$ is 3-connected and $u, v$ are distinct vertices in $G$. Show that there is a cycle *of even length* containing both $u$ and $v$.

A4 Let $G$ be a graph with chromatic number $k$ and let $c$ be a $k$-coloring of $G$. For each integer $i$ with $1 \leq i \leq k$, let $C_i$ denote the color class consisting of all vertices colored $i$. Prove that for each integer $i$ with $1 \leq i \leq k-1$, there exists a vertex in $C_k$ that is adjacent to a vertex in $C_i$.

A5 (a) Prove that if a connected cubic graph has a 1-factor $F$ and a bridge $e$, then $F$ must include the bridge $e$.

(b) Suppose that a connected cubic graph $G$ has a minimum edge-cut consisting of two edges $e$ and $f$. What can we conclude about any 1-factor of $G$? May it contain neither $e$ nor $f$? May it contain both $e$ and $f$? May it contain exactly one of $e$ and $f$?

A6 Let $H$ and $F$ be two cycle-free subgraphs of the same graph $G$, and assume that $H$ has more edges than $F$. Prove that there is an edge of $H$ that can be added to $F$ so that $F$ keeps its cycle-free property.

## Part B

B1 Let $G$ be a triangle-free graph of order $n$. Show that $\alpha(G) \geq \sqrt{n}$.

B2 Let $G$ be a bipartite graph with an even number of vertices, say $2n$ (note: this does not mean that the bipartition of $G$ is two equal sized sets). Prove that $\alpha(G) = n$ if and only if $G$ has a perfect matching.

B3 Prove the following.

(a) There exists only one 3-regular maximal planar graph.

(b) There exists only one 4-regular maximal planar graph.

(c) There exists no maximal planar graph $G$ with $\delta(G) = 3$ and $\Delta(G) = 4$ and an equal number of vertices of degree 3 and 4.

B4 Let $n$ be even and $k$ be odd. The $k$-regular graph $G$ is formed by arranging $n$ equally spaced vertices in a circle, and then connecting each vertex $v$ to the opposite vertex, as well as the $k-1$ nearest other vertices. Prove that $\kappa(G) = k$.

B5 Let $k \geq 2$ be an integer and let $n = \lfloor 3^{k/2} \rfloor$. Show that there exists a 3-coloring of the edges of $K_{n,n,n}$ such that there is *no* monochromatic and induced copy of $K_{k,k}$.

B6 Let $T$ be a tree and $P_n$ a path on $n$ vertices. Show that either

(i) there exists an edge $e \in E(T)$ such that $T - e$ contains no $P_n$, or

(ii) contains 3 vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.