Graph Theory Preliminary Examination

June 7, 2017

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which problem in Part A and which two problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Recall that a graph G is *k-critical* if $\chi(G) = k$ and $\chi(H) < \chi(G)$ for every proper subgraph H of G. Show that there is no 3-chromatic graph G such that $G - v$ is 3-critical for every vertex v of G.

A2 Let k be a positive integer and let G be a graph with at least $k+1$ vertices such that $\deg(u) + \deg(v) \geq n + k - 2$ for every two nonadjacent vertices u and v of G. Prove that G is k-connected.

A3 Recall that we say that G is *maximally planar* if G is planar, but adding any edge to G results in a nonplanar graph. Let G be a maximally planar graph such that $\chi(G) \leq 3$. Prove that G has a closed Eulerian trail.

A4 Let T be a tree. Prove that T has a perfect matching if and only if for every vertex $v \in V(T)$, $T - v$ has exactly one odd component.

A5 Recall that we say a graph G is *complete r-partite* if the vertex set $V(G)$ can be written as the union $V_1 \cup V_2 \cup \ldots \cup V_r$ of disjoint sets such that any two vertices u and v are adjacent if and only if they are in different sets V_i and V_j. We say G is *complete multipartite* if there exists some r such that G is complete r-partite.

Prove that G is a complete multipartite graph if and only if there is no set S of three vertices such that the induced graph $G[S]$ has exactly one edge.

A6 (i) State the regularity lemma. (Define all necessary notations.)

(ii) Let G be a bipartite graph with vertex classes X and Y such that $|X| = |Y| = n$. Suppose also that the maximum degree of G is at most $\varepsilon^2 n$, where $0 < \varepsilon < 1$. Show that the pair (X, Y) is ε-regular in G.

Part B

B1 Recall that we say that G is *claw-free* if G does not have $K_{1,3}$ as an **induced** subgraph. Suppose G is claw-free and has a proper k-coloring $c : V(G) \to \{1, \dots, k\}$.

(i) Prove that if the color classes are $V_1, \dots V_r$ (so $V_i = \{v \in V(G) : c(v) = i\}$), then the connected components of $G[V_i \cup V_j]$ are all paths and cycles.

(ii) Prove that there exists some proper k-coloring $c' : V(G) \to \{1, \dots, k\}$ such that any two color classes V_i' and V_j' differ in size by at most 1.

B2 Let $BR(k)$ be a *bipartite Ramsey number* which is the smallest number n such that in any 2-coloring of the edges of the bipartite graph $K_{n,n}$ there is a monochromatic copy of $K_{k,k}$. Show that for any $k \geq 2$ we have

$$BR(k) \geq 2^{k/2}.$$

Hint: Use a random coloring.

B3 We call a finite graph *minimally k-matchable* if it has at least k distinct perfect matchings but deleting any edge results in a graph which has not. Characterize (that means describe the structure) all minimally 2-matchable graphs.

B4 Let $G = (V, E)$ be a graph of order n with average degree d. We have seen in class that the independence number $\alpha(G)$ satisfies the following lower bound:

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}. \tag{1}$$

(i) Use (1) and the Cauchy-Schwarz inequality, $(\sum_i a_i)(\sum_i b_i) \geq (\sum_i a_i b_i)^2$, to show that

$$\alpha(G) \geq \frac{n}{d+1}. \tag{2}$$

(ii) Use (2) to show the Turán theorem, which asserts that any K_{k+1}-free graph of order n has at most $\left(1 - \frac{1}{k}\right) \frac{n^2}{2}$ edges.

B5 (i) Show that there exists no maximal planar graph containing only vertices of degree 3 and degree 5 and having an equal number of each.

(ii) Determine all nonplanar graphs G with $n \geq 5$ vertices and $m = 3n - 5$ edges having the property that for each edge e of G, the graph $G - e$ is planar.

B6 Recall that we say a graph G is *minimally 2-connected* if G is 2-connected but $G - e$ is not 2-connected for any edge $e \in E(G)$. Suppose G is minimally 2-connected.

(i) Prove that G has a vertex of degree 2.

(ii) Prove that if G has at least 4 vertices then $|E(G)| \leq 2|V(G)| - 4$.