Graph Theory Preliminary Examination

June 1, 2019

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in solutions to **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Let G be a connected planar graph of order $n \geq 5$ such that G is decomposed into three spanning subgraphs G_1, G_2, and G_3 (that is, the edge set $E(G)$ of G is partitioned into the three sets $E(G_1)$, $E(G_2)$, and $E(G_3)$). Prove that if two of the subgraphs G_1, G_2, and G_3 are trees, then the third subgraph is not a tree.

A2 Let G be a graph of order n and size m having connectivity $\kappa(G) = k \geq 3$ where $k \leq n-2$ and diameter $\text{diam}(G) = d \geq 3$.

(a) Prove for each integer j with $k \leq j \leq n-2$ that there exists a set S_j of j vertices of G such that $G - S_j$ is disconnected.

(b) Prove that there exist two vertices u and v of G and a collection W of internally disjoint $u-v$ paths such that at most $m - kd$ edges of G lie on no path belonging to W.

A3 Let G be a k-connected graph of order n where $3 \leq k \leq n-2$.

(a) Prove for every set S of k vertices of G that the graph G contains a tree T whose set of end-vertices is S.

(b) Is the statement in (a) true if G is a $(k-1)$-connected graph of order at least $k+2$ for any integer $k \geq 3$?

A4 A set Y of edges of a graph G *covers* all vertices in a set X of vertices of G if every vertex in X is incident with some edge in Y. A bipartite graph G has bipartition $A \cup B$ (namely the partite sets of G are A and B). Let D be the set of all vertices of G that have maximum degree $\Delta(G)$. Prove that there is a matching M of G that covers all the vertices in $D \cap A$.

A5 A set X of vertices of a graph G is a *vertex cover* if every edge of G is incident with at least one vertex in X. Let $\beta(G)$ be the vertex covering number of G (the smallest cardinality of a vertex cover in G) and let $\alpha(G)$ be the vertex independence number of G. By Gallai's theorem, if G is a graph of order n, then $\alpha(G) + \beta(G) = n$.

(a) Prove that if G is a triangle-free graph of size m, then $m \leq \alpha(G)\beta(G)$.

Hint: Show that G has a vertex of degree at least $m/\beta(G)$.

(b) Use (a) to prove Mantel's theorem:

If G is a triangle-free graph of order n and size m, then $m \leq n^2/4$.

Part B

B1 Let T be a tournament such that every vertex of T belongs to a directed cycle of T. Show that if u and v are two vertices of T for which $|\operatorname{od} u - \operatorname{od} v| \leq 1$, then u and v belong to the same strong component of T.

B2 While the Ramsey number $R(K_6, K_6)$ is unknown, it is known that it exists, that is, $R(K_6, K_6) = k$ for some integer k. Prove that if every edge of K_k is colored with one of the three colors red, yellow, and green, then there is either a red K_6, a yellow K_3, or a green K_3.

B3 (a) Let G be a connected graph of order 4 or more. Prove that if G contains both even vertices and odd vertices, then G contains adjacent vertices u and v such that u is odd and v is even.

(b) Determine, for each integer $n \geq 4$, the number of connected graphs G of order n with the property that $G - v$ is Eulerian for every vertex v of G.

B4 For a graph G of order n, let $\sigma_2(G)$ denote the minimum degree sum of two nonadjacent vertices of G. Recall the following for a graph G of order $n \geq 4$.

(1) If $\sigma_2(G) \geq n$, then G is Hamiltonian.

(2) If $\sigma_2(G) \geq n + 1$, then G is Hamiltonian-connected.

(3) If G is Hamiltonian-connected, then G is 3-connected and so $\delta(G) \geq 3$.

A graph G of order $n \geq 3$ is *k-leaf connected* for an integer k with $2 \leq k \leq n - 1$ if for every set S of k vertices, there exists a spanning tree T of G such that S is the set of end-vertices (leaves) of T. Therefore, 2-leaf connected and Hamiltonian-connected are the same concept.

(a) Prove that if G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n + 2$, then $G - w$ is Hamiltonian-connected for every vertex w of G.

(b) Prove that if G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n + 2$, then G is 3-leaf connected.

B5 For a graph G, let $\omega(G)$ denote the clique number of G (the order of a maximum clique of G) and let $\chi(G)$ be the chromatic number of G. Let G be a nontrivial connected graph that does not contain $2K_2 = K_2 + K_2$ as an **induced** subgraph and let K be a maximum clique in G where $V(K) = \{v_1, v_2, \ldots, v_\omega\}$ and $\omega = \omega(G) \geq 2$. For each vertex u of G that does not belong to K, define the color $c(u)$ of u as follows. If u is not adjacent to exactly one vertex of K, say v_i, then define $c(u) = i$. If u is not adjacent to two (or more) vertices of K, say v_i and v_j, then define $c(u) = \{i, j\}$.

(a) Show that this coloring c can be extended to a proper vertex coloring of G.

(b) Use (a) to show that $\chi(G) \leq \omega(G) + \binom{\omega(G)}{2}$.