Graph Theory Preliminary Examination

May 29, 2021

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in solutions to **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Let T be a strong tournament of order $n \geq 3$ and let D be a digraph obtained from T by adding a new vertex v to T, joining v to each vertex of T, and directing the edges incident with v in such a way that the indegree and outdegree of v are both positive. Show that D is Hamiltonian.

A2 Let G be a bipartite graph with partite sets (bipartition) A and B where $|A| = |B| = n$. Suppose that $\delta(G) \geq \frac{n}{2}$. Use Hall's theorem to prove that G has a perfect matching.

A3 Let G be a 3-regular graph with chromatic index $\chi'(G) = 3$. Suppose that the partition of $E(G)$ into three color classes is unique (in other words, there is a proper 3-coloring of the edges of G and this 3-coloring is essentially unique in the sense that the only way to get another proper 3-coloring is to permute the colors). Prove that G is Hamiltonian.

A4 It is well known that the Ramsey number $R(3,3) = 6$ and the Ramsey number $R(4,4) = 18$. However, is it also known that for $k \geq 5$, the Ramsey number $R(k,k)$ exists but is not known. Suppose that $R(6,6) = p$. Prove that for every red-green-yellow coloring of the edges of K_p, there is a red K_6, or a green K_3, or a yellow K_3.

A5 The graph C_5 has order 5, clique number 2, and chromatic number 3. The wheel $C_5 \vee K_1$ has order 6, clique number 3, and chromatic number 4. In fact, for $n \geq 7$, the graph $C_5 \vee K_{n-5}$ has order n, clique number $n-3$, and chromatic number $n-2$. So, $\chi(C_5 \vee K_{n-5}) = \omega(C_5 \vee K_{n-5}) + 1$. Prove that if G is a graph of order $n \geq 5$ with clique number $\omega(G) = n-2$, then $\chi(G) = n-2$.

Part B

B1 Let G be a graph. Show that $\chi(G) \leq k$ if and only if G has an orientation with no directed path of length k.

B2 A graph G of order 10 and size 30 is 2-cell embedded on the torus. Answer the following questions, each with a brief explanation.

(a) How many regions are there in this embedding?

(b) Can G be embedded in the plane?

(c) What is the genus $\gamma(G)$ of G?

(d) What is the genus $\gamma(G+uv)$ of $G+uv$, where u and v are nonadjacent vertices in G?

B3 Prove, for every positive integer k, that the complete graph K_{6k+4} is 3-factorable, where each 3-factor is Hamiltonian.

B4 Let G be a graph with an edge xy such that x and y have degree at least k. Suppose that G/xy (formed by contracting the edge xy) is k-connected. Prove that G is also k-connected.

B5 For the path P_{d+1} of diameter d, its domination number is $\gamma(P_{d+1}) = \left\lceil \frac{d}{2} \right\rceil$. Thus, for every connected graph G of diameter d, $\gamma(G) \geq \left\lceil \frac{d}{2} \right\rceil$. It has been shown that there exists a connected graph of diameter 2 having domination number 6. Whether there are connected graphs of diameter 2 having domination number greater than 6 appears to be unknown. Even the graph with diameter 2 and domination number 6 was not easy to verify.

Let $k \geq 4$ be an integer. A graph G of order k^2+1 is constructed as follows. A vertex v of G has degree k and has k neighbors are $v_1, v_2, \ldots, v_k$. For each integer i with $1 \leq i \leq k$, the vertex v_i is adjacent to $k-1$ vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,k-1}$ in addition to v. For $1 \leq j \leq k-1$, let $S_j = \{v_{i,j} : 1 \leq i \leq k\}$. In particular, $S_1 = \{v_{1,1}, v_{2,1}, \ldots, v_{k,1}\}$ and $S_{k-1} = \{v_{1,k-1}, v_{2,k-1}, \ldots, v_{k,k-1}\}$. Thus, $|S_j| = k$ for $1 \leq j \leq k-1$. We add an edge between every two vertices of S_j, that is, $G[S_j] = K_k$ for $1 \leq j \leq k-1$.

(a) Determine (with a brief explanation) the diameter of G.

(b) Determine (with explanation) the domination number of G.