WMU Department of Mathematics
Algebra Comprehensive Exam
August 26, 2019

Instructions. Work all of these problems and write your solutions on the paper provided. Start each solution at the top of a new sheet of paper. Write your name on every page. Please write clearly and completely. In considering how much detail to write, you are urged to err on the side of caution. Thus, for example, if you have doubts about whether you need to demonstrate how to prove a particular assertion, then it is likely better to do so. (Natually, you are always free to ask your examiners about this.) Unless otherwise arranged, you have 6 hours to complete this exam. You are not allowed to use books, notes, or a device capable of radio communication during this exam.

1. Let G be a group, and X a set on which G acts transitively. (a) Let $x \in X$, and let G_x be the stabilizer of x in G. Show that $X \cong G/G_x$ as G-sets. (b) Suppose that H is a subgroup of G which acts transitively on X. Show that $G = HG_x$.

2. Let R be a commutative ring with identity, m and n positive integers, and let $f : R^n \longrightarrow R^m$ be a surjective R-module homomorphism with kernel K. Show that $R^n \cong R^m \oplus K$.

3. Let G be a group of order 12 with $G \not\cong A_4$, then G has exactly two elements of order 3.

4. Let K be a field and let $K[[x]]$ denote the ring of formal power series in one variable over K.
   (a) Show that $K[[x]]$ is a local ring (i. e., it has a unique maximal ideal).
   (b) Describe the group of units $K[[x]]^*$ in terms of the coefficients of its elements. Prove your answer.
   (c) Assume that the characteristic of K is zero. Define the Fibonacci sequence of integers by:
   $$F_0 = F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2$$
   Define $f(x) \in K[[x]]$ by $f(x) = \sum_{n \geq 0} F_n x^n$. Show that $f(x)^{-1} = 1 - x - x^2$.

5. The **Heisenberg group** G over the field K is the subgroup of $GL_3(K)$ defined by matrices of the form
   $$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad (x, y, z \in K).$$
   Determine and describe the center $Z(G)$, the commutator subgroup $[G, G]$, and the torsion subgroup $\mathrm{Tor}(G)$ (the subgroup generated by the elements of finite order). (Note: Any or all of these may depend on the characteristic of K.)

6. (a) Prove that AB and BA have the same eigenvalues whenever A and B are $n \times n$ matrices over $\mathbb{C}$. (b) Must AB and BA always have the same Jordan form? Prove or give a counterexample.

7. Let ζ be a primitive 11^{th} root of unity and let $\alpha = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$. It is well known that $Gal(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/11\mathbb{Z})^\times$, where $k \mod 11$ corresponds to $\zeta \mapsto \zeta^k$.
   (a) Let S be a subset of $\{1, \zeta, \zeta^2, \ldots, \zeta^{10}\}$. Show that if S has 10 elements then S is linearly independent over $\mathbb{Q}$.
   (b) Show that $\alpha \notin \mathbb{Q}$.
   (c) Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.