WMU Department of Mathematics
Algebra Comprehensive Exam
Spring 2020

Instructions. Work all of these problems and write your solutions on the paper provided. Start each solution at the top of a new sheet of paper. Write your name on every page. Please write clearly and completely. In considering how much detail to write, you are urged to err on the side of caution. Thus, for example, if you have doubts about whether you need to demonstrate how to prove a particular assertion, then it is likely better to do so. (Natually, you are always free to ask your examiners about this.) Unless otherwise arranged, you have 6 hours to complete this exam. You are not allowed to use books, notes, or a device capable of radio communication during this exam.

1. Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$, and $I_3 = (3, 2+\sqrt{-5}) \subset R$.
   (a) Prove that I_3 is a prime ideal of R.
   (b) Show that the ideal (3) is not prime, but that 3 is irreducible.

2. Let F be a field. The general linear group $G = GL(n, F)$ consists of all invertible matrices over F. The group G acts, by left multiplication, on the vector space $V = M_n(F)$ of all $n \times n$ matrices over F. Explain in detail how the set of orbits of this group action is in one-one correspondence with the set of all linear subspaces of F^n. (Hint: Given A in V, consider the nullspace of A.)

3. Prove that no group of order 30 is simple. If you choose to resort to the so-called "*pqr*-Theorem", then you must prove the theorem.

4. Let K be a field having $q = p^\ell$ elements, where p is prime.
   (a) Define $f : K \to K$ by $f(x) = x^p$. Show that f is a bijective ring homomorphism.
   (b) Define $F = \{x \in K : f(x) = x\}$. Show that F is a subfield of K.
   (c) Define $\mathrm{Tr} : K \to K$ by $\mathrm{Tr}(x) = x + x^p + x^{p^2} + \cdots + x^{p^{\ell-1}}$. Show that $\mathrm{Tr}(x) \in F$ for all $x \in K$ and that $\mathrm{Tr} : K \to F$ is an F-linear transformation.
   (d) Show that Tr is not the zero transformation.

5. Let G be a group of order n. There is a homomorphism $G \longrightarrow S_n$, where $g \in G$ maps to the permutation π_g, where π_g permutes the n elements of G by $\pi_g(x) = gx$.
   (a) Show that π_g is an odd permutation if and only if g has even order and $[G : \langle g \rangle]$ is odd.
   (b) Show that if a Sylow 2-group of G is non-trivial and cyclic then G has a subgroup H with $[G : H] = 2$. (Hint: The even elements of S_n comprise a subgroup with index 2.)

6. Let R be a commutative ring with 1, and let A, B, C, D, E be R-modules. Suppose we have a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & E & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle id} & & \\ C & \xrightarrow{r} & D & \xrightarrow{s} & E & \longrightarrow & 0 \end{array}$$

   with exact rows.
   (a) Show that r induces a well-defined map $\overline{r} : C/\alpha(A) \longrightarrow D/\beta(B)$.
   (b) Show that $\overline{r}$ is surjective.