Analysis Prelim

February 24, 2001.

Do six of the following problems.

1. (a) Define what it means for a real-valued function f to be measurable.

(b) Prove or disprove: If f and g are two real-valued measurable functions, then $f + g$ is measurable.

2. Let $(X, \mathfrak{B}, \mu)$ be a measure space, and let $A, B \in \mathfrak{B}$. Define $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$, where $A \setminus B = \{x \in X : x \in A \text{ and } x \notin B\}$. Prove or disprove: $\mu(A \bigtriangleup B) = 0 \Rightarrow \mu A = \mu B$.

3. Let $\{f_n\}$ be a convergent sequence in $L^p(a, b)$. Prove or disprove: There exists a subsequence $\{f_{n_k}\}$ that converges almost everywhere.

4. (a) Define what it means for measures μ and ν to be mutually singular. Define what it means for a measure ν to be absolutely continuous with respect to a measure μ.

(b) Prove or disprove: If $\nu \ll \mu$ and $\nu \perp \mu$ then $\nu = 0$.

5. Let $\{f_n\}$ be a sequence in $L^p(a, b)$ for some $p > 1$. Suppose that f_n converges almost everywhere to a function $f \in L^p(a, b)$ and that $\|f_n\| \leq M$. Prove or disprove: For any function $g \in L^q(a, b)$, where $1/p + 1/q = 1$, $\lim \int f_n g = \int fg$.

6. Let f be an integrable function on $[-2, 2]$. Prove or disprove: The function $F(h) = \int_{-1}^{1} f(x + h)\, dx$ is continuous on $(-1, 1)$.

7. Let $g : X \to \mathbb{R}$ be a μ-integrable function and let $f : Y \to \mathbb{R}$ be a ν-integrable function. Define $f : X \times Y \to \mathbb{R}$ by $f(x, y) = g(x)h(y)$. Show that f is $\mu \times \nu$-integrable and that

$$\int_{X \times Y} f \, d(\mu \times \nu) = \left(\int_X g \, d\mu \right) \left(\int_Y h \, d\nu \right).$$

8. State and prove the most general version of the Fundamental Theorem of Calculus that you know.