Analysis Prelim

August 16, 2001.

Do six of the following problems.

1. (a) Define what it means for a collection $\mathcal{F}$ to be an algebra.

   (b) Define what it means for a collection $\mathcal{F}$ to be a σ-algebra.

   (c) Give an example of an algebra that is not a σ-algebra.

2. (a) Define what it means for a set $A \subset \mathbb{R}$ to be Lebesgue measurable.

   (b) Show that the interval $(0,1)$ is Lebesgue measurable.

3. Show that if $f \in L^1(A)$ and for any measurable set $B \subset A$

$$\int_B f(x)dx = 0$$

then $f(x) = 0$ almost everywhere on A.

4. Let E be a measurable set with finite measure and let $\{f_n\}$ be a sequence of nonnegative integrable functions satisfying $\int_E f_n \to 0$.

   (a) Show that the sequence $\{f_n\}$ converges to 0 in measure.

   (b) Give an example that shows that $\{f_n\}$ need not converge to 0 almost everywhere.

5. Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue integrable function and define the function $F : [0,1] \to \mathbb{R}$ by $F(t) = \int_0^1 f(x)\sin(xt)\,dx$.

   (a) Show that F is well defined.

   (b) Show that F is differentiable and $F'(t) = \int_0^1 xf(x)\cos(xt)\,dx$.

(c) Show that F has derivatives of all orders and that

$$F^{(2n)}(t) = (-1)^n \int_0^1 x^{2n} f(x) \sin(xt)\, dx$$

and

$$F^{(2n-1)}(t) = (-1)^{n+1} \int_0^1 x^{2n-1} f(x) \cos(xt)\, dx.$$

6. Let f be a bounded measurable function on $[0,1]$. Prove or disprove: $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

7. Let M be a set and ρ a function $\rho : M \times M \to \mathbb{R}$.

(a) Define what it means that a couple (M, ρ) is a complete metric space.

(b) Let $\rho(x,y) = |\arctan x - \arctan y|$. Prove or disprove: $(\mathbb{R}, \rho)$ is a complete metric space.

8. Let V be a vector space, let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two inner products on V, and let $\| \cdot \|_1$ and $\| \cdot \|_2$ be the norms corresponding to these inner products. Suppose that V is complete in both norms and that it is separable in each of the norm-toplogies. Show that the normed spaces $(V, \| \cdot \|_1)$ and $(V, \| \cdot \|_2)$ are isomorphic.

9. Let m denote the Lebesgue measure on $\mathbb{R}$ and let ν be a measure defined as $\nu(A) = m(A \cap [-2, 2])$ for every Lebesgue measurable set A. Prove that ν is absolutely continuous with respect to m and find explicitly the Radon-Nikodym derivative $d\nu/dm$.