August 21, 2002

ANALYSIS PRELIMINARY EXAMINATION

(ANSWER ANY FIVE OF THE FOLLOWING EIGHT QUESTIONS)

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1. Define what it means for a real-valued function f to be measurable. Prove or disprove: If f and g are two real-valued measurable functions then
(a) $f \cdot g$ is measurable;
(b) function $h(x) = \min\{f(x), g(x)\}$ is measurable.
(c) function $(f(x))^{g(x)}$ is measurable if $f(x) > 0$.

2. Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be absolutely continuous. Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that

$$|g(x) - g(y)| \leq L|x - y| \quad \forall\ x,\ y$$

where L is some constant. Show that for any absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ the composition $g \circ f$ is absolutely continuous.

3. Let (X, μ) be a finite nonnegative measure space. Define what it means for a sequence of real-valued measurable functions $\{f_n\}$ to converge in measure. Prove that f_n converges in measure to f if and only if

$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu = 0$$

4. Let (X, μ) and (Y, ν) be complete nonnegative measure spaces such that $\mu(X) = \nu(Y) = 1$. For a set $E \subset X \times Y$ and $x \in X$ the set $E_x \subset Y$ is defined as follows

$$E_x := \{y \in Y \mid (x, y) \in E\}$$

Prove that if for the set $E \subset X \times Y$

$$(\mu \times \nu)(E) = 1,$$

then $\nu(E_x) = 1$ almost everywhere on X.

5. Let (X, μ) be a finite nonnegative measure space and $f : X \to \mathbb{R}$ be measurable. Prove that f is integrable if and only if

$$\sum_{k=1}^{\infty} k\mu(\{x \in X \mid k \leq |f(x)| < k + 1\}) < +\infty.$$

6. Using the definition of Riemann integral and relating limit to integration, evaluate the following limit

$$\lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n+1} + \cdots \frac{1}{2n-1})$$

7. Let F be a linear functional on a normed space X. Prove or disprove: F is continuous if and only if it is bounded.

8. Let $f : [a, b] \to R$ be Lebesgue integrable. Prove that there exists constant c_0 such that

$$\inf_c \int_{[a,b]} |f(x) - c| dx = \int_{[a,b]} |f(x) - c_0| dx.$$

Prove that such c_0 is unique if the function f is continuous.