February 28, 2003

ANALYSIS PRELIMINARY EXAMINATION

(ANSWER ANY FIVE OF THE FOLLOWING EIGHT QUESTIONS)

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1. Let x_n be a bounded sequence of elements in a separable Hilbert space $\mathbb{H}$. show that there exists a subsequence x_{n_k} which converges weakly.

2. (a) Give the definition of a measurable function $f : \mathbb{R} \to \mathbb{R}$. (b)Let f_n be a sequence of measurable functions on a measurable set E. Prove or disprove: the function $f = \limsup_{n\to\infty} f_n$ is measurable.

3. Let f_n be a sequence of Lebesgue measurable functions on $[0, 1]$.

(a) Define convergence in L^p and in measure for f_n to a function f on $[0, 1]$.

(b) Prove or disprove: The sequence f_n converges in L^p, $p \geq 1$, to f implies that f_n converges in measure to f.

4. Let $(\mathbb{X}, \mu)$ be a measure space, $f \in L^1(\mathbb{X}, \mu)$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set E satisfying $\mu(E) < \delta$ the following holds

$$\int_E |f| d\mu < \varepsilon$$

5. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of measurable functions defined on $E \subset \mathbb{R}$ such that $\mu(E) < \infty$. Suppose that f_n converges to f in measure and g_n converges to g in measure. Prove or disprove: $f_n g_n$ converges in measure to fg. Consider the same problem when $\mu(E) = \infty$.

6. Let f be a continuous real-valued function, and let g be a (Lebesgue) measurable real-valued function, both defined on $\mathbb{R}$. Prove or disprove: the function $h(x) = g(f(x))$ is (Lebesgue) measurable.

7. Compute $\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx.$

8. If f_n is a sequence in $L^1(\mathbb{X}, \mu)$ which converges uniformly on $\mathbb{X}$ to a function f, and if $\mu(\mathbb{X}) < +\infty$, then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

7) Let $1 \leq p \leq \infty$. Suppose that $\{f_n\}$ is a sequence in $L^P[0,1]$ and that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

(a) Show that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e.

(b) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. Show that $f \in L^p[0,1]$ and that $\|f\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p$.

8) Prove or disprove that a finite valued linear functional on a normed space X is discontinuous if and only if $F(\{x \in X : \|x - a\| < r\}) = \mathbb{R}$ for any $a \in X$ and any $r > 0$.