Analysis Comprehensive Examination

March 4, 2004

Do six of the following eight problems.

1) Let E be a Lebesgue measurable set and let f_n be a sequence of Lebesgue measurable funtions. Prove or disprove the following:

(a) If f_n conversges to 0 in measure, then

$$\int_E \frac{|f_n|}{1+|f_n|}\, d\mu \to 0.$$

(b) If

$$\int_E \frac{|f_n|}{1+|f_n|}\, d\mu \to 0,$$

then f_n converges to 0 in measure on E.

2) Given an $\alpha > 0$ a function $f : (a,b) \to \mathbb{R}$ is *Lipschitz of order* α if there is a $C > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x, y \in (a,b)$.

(a) Assume that f is Lipschitz of order $\alpha > 1$. Prove that $f'(x) = 0$ for all $x \in (a,b)$.

(b) Prove that if f is Lipschitz of order 1 on (a,b), then f' exists almost everywhere and is uniformly bounded.

3) Let A denote the set of all $x = \{\psi_n\}$ in ℓ_2 such that $\sum_{n=1}^{\infty} \psi_n = 0$. Prove or disprove that A is dense in ℓ_2.

4) Let X be a real Banach space.

(a) Show that any sequence $\{x_n^*\}$ that converges strongly in X^* converges in the weak* topology.

(b) Show that the weak* topology is Hausdorff.

(c) Show that any weakly convergent sequence $\{x_n^*\}$ in X^* is weak* convergent in X^*.

5) Let m^* be Lebesgue outer measure. Prove that a bounded set E is Lebesgue measurable if and only if for every $\varepsilon > 0$ there is a a closed subset F of E such that $m^*(E - F) < \varepsilon$.

6) Consider a positive measure space (X, μ).

(a) State and prove Fatou's Lemma.

(b) Does this theorem hold for general measurable functions? Explain.

7) Let $1 \leq p \leq \infty$. Suppose that $\{f_n\}$ is a sequence in $L^P 0, 1$ and that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

(a) Show that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e.

(b) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. Show that $f \in L^p[0, 1]$ and that $\|f\|_p = \sum_{n=1}^{\infty} \|f_n\|_p$.

8) Prove or disprove that a bounded linear functional on a normed space X is discontinuous if and only if $F(\{x \in X : \|x - a\| < r\}) = \mathbb{R}$ for any $a \in X$ and any $r > 0$.