Analysis Preliminary Examination

SUMMER 2006

(Solve any six of the following eight problems)

1. (a) Let $\{a_n\}$ and $\{b_n\}$ be real sequences. Prove

$$\text{(i)} \quad \limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided that the right side in (i) is not $\infty - \infty$;

$$(ii) \quad \limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n)$$

.

(b) Show that for a sequence of measurable functions $f_n(x)$ the following functions

$$\sup_n f_n(x)$$
$$\limsup_{n\to\infty} f_n(x)$$

are measurable.

2. (a) Prove that a continuous function on a finite closed interval attains its minimum.

(b) Suppose f is differentiable on a finite interval (a, b) and

$$\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x) = +\infty.$$

Prove that, for any $r \in R$, there exists a $c \in (a, b)$ such that $f'(c) = r$.

3. (a) Show that, for any $y \in (0, +\infty)$, the function

$$f(y) = \int_0^{+\infty} e^{-yx}\frac{\sin x}{x}dx$$

is well defined.

(b) Prove that $f(y) + arctan(y)$ is a constant on $(0, +\infty)$.

(c) Use the result in (b) to show that

$$\int_0^{+\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

4. Suppose every point in the interval $[0, 1]$ belongs to at least k sets of the family of Lebesgue measurable sets $E_1, E_2, \ldots, E_N$. Prove that for some $n \in \{1, \ldots, N\}$ the following inequality holds

$$\lambda(E_n) \ge \frac{k}{N}$$

where $\lambda(A)$ denotes Lebesgue measure of the set A. (Hint: use characteristic functions of sets E_n.)

5. (a) State the definition of an absolutely continuous function $f : [a, b] \to \mathbb{R}$.

(b) Let $f(x)$ be an absolutely continuous function on $[a, b]$. Prove or disprove: f can be written as a difference of two strictly increasing absolutely continuous functions.

6. (a) Give the definition of a linear bounded functional $f : \mathbb{X} \to \mathbb{R}$ on Banach space $\mathbb{X}$.

(b) Let $C[0, 1]$ be the space of all continuous functions on $[0, 1]$ with the norm $\|g\| = \max_{t\in[0,1]} |g(t)|$. Let $h \in L^1[0, 1]$ and consider the following functional on $C[0, 1]$

$$f(g) := \int_{[0,1]} h(t)g(t)dt$$

Show that f is a bounded linear functional on $C[0, 1]$ and find its norm.

7. (a) Give the definition of the space $L^p[0, 1]$ of functions $f : [0, 1] \to \mathbb{R}$.

(b) State Holder inequality.

(c) Suppose $f \in L^p[0, 1]$ with $p > 1$. Show that $f \in L^s[0, 1]$ for any $1 \leq s < p$.

(d) Suppose that a sequence of functions $f_n \in L^p[0, 1]$ converges to f in $L^p[0, 1]$. Show that for any $g \in L^q[0, 1]$ with q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\lim_{n\to\infty} \int_{[0,1]} g f_n dx = \int_{[0,1]} g f dx$$

8. (a) State Monotone Convergence Theorem.

(b) Use this Theorem to show that for any integrable nonnegative function $f : [a, b] \to \mathbb{R}$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set A satisfying $m(A) < \delta$

$$\int_A f(x)dx < \varepsilon$$

(*Hint:* Consider the sequence $f_n(x) := \min\{f(x), n\}$ and $\lim_{n\to\infty} \int f_n dx$)