Analysis Comprehensive Exam

Do 5 of the following problems. March 8, 2007

This is a 3 hour exam.

1) Let f_n be a sequence in $L^2[0,1]$ and let $f \in L^2[0,1]$.

(a) Give definitions of f_n converges to $f \in L^2[0,1]$ in mean and weakly.

(b) Prove that if f_n converges weakly to f and $\lim_{n\to\infty} \|f_n\|_2 = \|f\|_2$ then f_n converges to f in mean.

(c) Is the converse of (b) true? Justify your answer.

2) Let $f : (0,1) \to R$ be a Lebesgue measurable function bounded almost everywhere. Show that there exists a unique number r such that

(a) $\mu(\{x \in (0,1) : f(x) \geq r\}) \geq 1/2$, and

(b) for any $s > r$, $\mu(\{x \in (0,1) : f(x) \geq s\}) < 1/2$,

3) An extended real valued function M on sets is subadditive if $M(A \cup B) \leq M(A) + M(B)$.

(a) Show that outer measure is subadditive on the subsets of $\mathbb{R}$.

(b) Fix a bounded interval $I \subset \mathbb{R}$ and define the inner measure of $A \subset I$ as

$$m_*(A) = \sup_{\cup I_n \subset A} l(I_n).$$

Here each I_n is an open interval contained in I. Is m_* a subadditive function on the subsets of I?

(c) Give a class of measurable subsets of a nontrivial bounded interval I whose inner measures do not equal their outer measures.

4) Consider the set $E = \{(x,y) : y \geq x,\ \ x,y \in [0,1]\}$. Recall that a base for the σ-algebra of product measurable subsets of $B = [0,1] \times [0,1]$ is the set $\mathcal{C}$ of products of intervals $I \times J$ where I and J are intervals contained in $[0,1]$.

(a) Show that E is a $F_{\sigma\delta}$ of elements in $\mathcal{C}$.

(b) Use this to show that for any Lebesgue integrable function $f(x,y)$ on B with the product measure, f is integrable on E and

$$\iint_E f \, d(\mu \times \mu) = \int_{[0,1]} \int_{[x,1]} f(x,y) \, d\mu(y) \, d\mu(x).$$

5) Let $f : [a,b] \to R$ be integrable and for every $c \in [a,b]$

$$\int_{[a,c]} f(x)dx = 0.$$

Prove that $f(x) = 0$ for almost all $x \in [a,b]$.

6)

(a) State a definition of *convergence in measure* for a sequence of measurable functions $\{f_n\}_n$.

(b) Let a sequence $\{f_n\}_n$ of functions in $L^2[0,1]$ converge to f in measure and assume that for some constant K

$$\|f_n\| \leq K \qquad \forall\ n$$

Show that $\{f_n\}_n$ converges to f weakly.

7)

(a) State a definition of continuity for functions $f : [a,b] \to R$

(b) Show that f is continuous on $[a,b]$ if and only if for any $r \in R$ the sets $\{x \in [a,b] : f(x) \geq r\}$ and $\{x \in [a,b] : f(x) \leq r\}$ are closed.

8) Let X be a linear space and let $\|\cdot\|$ and $\|\cdot\|_1$ be two norms on X so that both $\big(X, \|\cdot\|\big)$ and $\big(X, \|\cdot\|_1\big)$ are Banach spaces. Suppose that there is a positive constant α such that $\|x\| \leq \alpha\|x\|_1$ for all $x \in X$. Prove or disprove that there is a constant $\beta > 0$ such that $\|x\|_1 \leq \beta\|x\|$ for all $x \in X$.

9) Let $f(x)$ be an absolutely continuous function on $[a,b]$. Prove or disprove that $f(x)$ can be written as the difference of two strictly increasing absolutely continuous functions.