Anaylsis Comprehensive Exam

Summer 2008

Do as many of the following as you can.

1) Let X and Y be two finite measure spaces and let Z be their product with the standard product measure.

  1) Let $C \subset X$ be measurable and let $D \subset Y$ be non-measurable. Prove that $C \times D$ is not measurable.

  2) Prove or disprove: If E is a subset of Z such that its projection onto X has positive measurable and its projection on Y is not measurable, then E is not measurable.

  3) Let X and Y both be $[0, 1]$ with Lebesgue measure. Give an example of a set in Z that is measurable, but both of its projections are not measurable.

2) State and prove the Heine-Borel theorem.

3) Let $X = \cup_{n=1}^{\infty} X_n$ and let $\mathcal{M}_n$ be a σ-algebra on $X_n, n = 1, 2, \ldots$. Show that if $\mathcal{M} = \{A \subset X : A \cap X_n \in \mathcal{M}_n \text{ for all } n\}$, then $\mathcal{M}$ is a σ-algebra.

4) Give an example of a sequence of functions in $L_1[0, 1]$ which converges to 0 a.e., but does not converge in the L_1 norm.

5) Let A denote the set of all $x = (\xi_n)$ in ℓ_2 such that $\sum_{n=1}^{\infty} \xi_n = 0$. Prove or disprove that A is dense in ℓ_2.

6) Let $E \subset R$ be a Lebesgue measurable set.

  (a) Give the definition of a function $f : E \to R$ to be *Lebesgue measurable*.

  (b) Directly prove the following theorem: Let f_n be a sequence of Lebesgue measurable functions on E such that f_n converges to f almost everywhere. Then f is a Lebesgue measurable function.

  (c) Show that if $f : R \to R$ is differentiable everywhere then f' is Lebesgue measurable.

7)

  (a) State the definition of *convergence in measure*.

  (b) Let f_n be a sequence of measurable functions. Show that f_n converges to a measurable function f in L_2 implies that f_n converges to f in measure.