WMU Anaylsis Comprehensive Exam

Spring 2009

Do six of the following problems.

1) Let μ and ν be two measures on X. Show that $\mu + \nu$ is a measure on X.

2) Let $(X, \mathcal{M}, \mu)$ be a measure space and let A be a dense set in $\mathbb{R}$. Show that a function f on X is measurable if and only if for each $a \in A$ the set $f^{-1}\big([a, \infty)\big)$ is measurable.

3) Show that a linear functional $\mathcal{L}$ on a normed space is continuous if and only if $\mathcal{L}^{-1}\big(0\big)$ is closed.

4) Let $(X, \mathcal{M}, \mu)$ be a complete finite measure space and let f be a nonnegative measurable function on $(X, \mathcal{M}, \mu)$. Consider $X \times \mathbb{R}$ with the product measure of μ with Lebesgue measure.

(a) Show that the set $M = \{(x, \alpha) | 0 \le \alpha \le f(x)\}$ is measurable.

(b) Relate the measure of M to the integral of f on X.

5) Let f be an integrable function that is positive almost everywhere. If A is a set such that $\int_A f\, dm = 0$, then $m(A) = 0$.

6) Consider a measure space $(Y, \mathcal{Y}, \nu)$ such that for all $A \in \mathcal{Y}$, $\nu(A) \in \{a_1, \ldots, a_k\}$ for some integer k.

(a) Describe the measurable sets if there are exactly four values for $\nu(A)$.

(b) Can there be exactly six values for $\nu(A)$? Explain your answer.

7) Show that $L^\infty([0, 1])$ with Lebesgue measure is not separable.

8) Demonstrate that if a measure space $(X, \mathcal{B}, \mu)$ is not σ-finite, then the Radon-Nikodym theorem does not hold.