Analysis Comprehensive Examination

September 1, 2010

Do 5 of the following 8 problems. Only 5 problems will be graded; if a candidate submits more than 5 problems, those with a smaller number will be considered.

Every problem (1 thru 8) is graded on the 0–3 scale: 0 means very little work in the right direction has been submitted; 1 means that some progress in the right direction was made; 2 means that a small portion of the work is missing or is incorrect; 3 means that the solution is complete, with a couple of minor errors. Thus the maximum number of points that one can obtain is 15. In order to pass a candidate needs 10 or more points.

1. (a) Prove that a differentiable function with bounded derivative on a closed bounded interval $[a, b]$ is of bounded variation.
(b) Prove or disprove: Any function that is differentiable on a closed bounded interval $[a, b]$ is of bounded variation on $[a, b]$.

2. Prove that a set $E \subset \mathbb{R}$ is disconnected if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ whose range is a discrete set with at least two elements.

3. (a) Prove or disprove: if f and g are absolutely continuous then $f \cdot g$ is absolutely continuous.
(b) Prove or disprove: if f is absolutely continuous and g is continuous then $f \cdot g$ is absolutely continuous.

4. Let for $1 \le p < +\infty$ a sequence $f_n \in L_p(X)$ converges to f in measure and there exists a function $g \in L_p(X)$ such that $|f_n(x)| \le g(x)$ for a.a. $x \in X$. Show that f_n converges to f in $L_p(X)$.

5. Let (X, B, μ) be a measurable space. Show that the function $\rho(A, B) := \mu(A\Delta B)$ determines a quasi-metric on the space B, that is a non-negative symmetric function satisfying to the triangle inequality.

6. Let $\mathfrak{B}$ denote the algebra of all Borel sets in $\mathbb{R}$, let $\mathfrak{B}_2$ denote the algebra of all Borel sets in $\mathbb{R}^2$, and let $\mathfrak{B} \times \mathfrak{B}$ denote the minimal algebra that contains all sets $\{E \times F : E, F \in \mathfrak{B}\}$. Prove that $\mathfrak{B} \times \mathfrak{B} = \mathfrak{B}_2$.

7. Let m be Lebesgue measure on $[0, 1]$ and let $F = [0, 1/2]$. Define $f_1(x) = 2\chi_F(x) - 1$, $f_2(x) = x$, and $\mu_i(x) = \int_E f_i(x)\, dm$, $i = 1, 2$. Prove or disprove: $\mu_1 \ll \mu_2$.

8. Let h be a function in $L^\infty([0, 1])$ and let $\alpha(f) = \int_0^1 f(t)\, h(t)\, dt$, for $f \in L^1([0, 1])$. Show that α is a bounded linear functional on $L^1([0, 1])$ and find its norm.