Analysis Prelim

August 28, 2012

Solve any 5 of the next 8 problems.

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n, f \in L^p(\mu)$.
(a) State the definition of convergence of f_n to f in measure and the definition of convergence of f_n to f in L^p, respectively.
(b) Prove or disprove: f_n converges to f in L^p implies that f_n converges to f in measure.

2. For a sequence of functions $f_n : X \to R$ prove or disprove:

$$\{x \in X : \liminf_{n\to\infty} f_n > a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x \in X : f_k > a + \frac{1}{m}\}.$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $g_k : \mathbb{R} \to \mathbb{R}$, $1 \le k \le n$, be measurable functions. Prove that $h(x) := f(g_1(x), g_2(x), \ldots, g_n(x))$ is a measurable function.

4. Let T_n be a sequence of bounded linear functionals on $L^p, p > 1$. Suppose that for every $f \in L^p$, the sequence $T_n(f)$ is convergent, and let T be a functional on L^p defined by $T(f) = \lim_{n\to\infty} T_n(f)$. Prove that T is a bounded linear functional on L^p.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space.
(a) State Fatou's Lemma.
(b) Let $f_n, n = 1, \ldots$ be a sequence of non negative integrable functions in $L^1(\mu)$ such that $f_n \to f$ a.e. on X and

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu. \tag{0.1}$$

If $E \subset X$ is a measurable set, then show that

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$$

6. (a) State Holder inequality for functions $f \in L^p$ and $g \in L^q$.
(b) Let $f \in L^2[0,1]$ and it satisfies

$$\int_0^1 x^n f(x) dx = 0 \text{ for each } n = 0, 1, 2, ...,$$

Show that $f = 0$ a.e.

7. Let $f : [0, +\infty) \to R$ be a continuous function such that $\lim_{x\to+\infty} f(x) = \gamma$. Show that for any $a > 0$

$$\lim_{n\to\infty} \int_0^a f(nx) dx = a\gamma$$

8. Consider the mapping $T : C[0,1] \to C[0,1]$ defined by $Tf(x) = x^3 f(x)$ for all $f \in C[0,1]$ and each $x \in [0,1]$.
(a) Show that T is a bounded linear operator.
(b) Let $I : C[0,1] \to C[0,1]$ denotes the identity operator i.e., $I(f) = f$ for each $f \in C[0,1]$), then show that $\|I + T\| = 1 + \|T\|$.