2013 Analysis Preliminary Examination

Select 5 from the following 7 problems

1. Let A, B be any sets with power sets $\mathbf{P}(A)$ and $\mathbf{P}(B)$, let $\mathcal{A} \subset \mathbf{P}(A)$ and $\mathcal{B} \subset \mathbf{P}(B)$, and let $f : A \to B$. Define $f(\mathcal{A}) = \{f(X) : X \in \mathcal{A}\}$ and $f^{-1}(\mathcal{B}) = \{f^{-1}(Y) : Y \in \mathcal{B}\}$.

(a) If $\mathcal{B}$ is a σ-algebra, prove that $f^{-1}(\mathcal{B})$ is also a σ-algebra.

(b) Give an example to show that if $\mathcal{A}$ is a σ-algebra, $f(\mathcal{A})$ need not be a σ-algebra.

2. Let $f : R \to R$ be continuous and $g : [a, b] \to R$ be absolutely continuous.

(a) Sow that the function

$$F(x) := \int_0^{g(x)} f(t)dt$$

is absolutely continuous.

(b) Find

$$\frac{d}{dx}F(x)$$

for almost all $x \in [a, b]$.

3. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $f : X \to R$ be a measurable function.

(a) Show that if for each natural n function f^n is integrable and $\lim_{n\to\infty} \int f^n d\mu$ exists then

$$|f(x)| \leq 1 \quad \text{for almost all } x \in X$$

(b) If f^n is integrable for each n and $\int f^n d\mu = c$ for some constant c then show that $f(x) = \chi_A(x)$ for some measurable set $A \subset X$.

4. (a) State the Baire Category Theorem.

(b) Let $f : R^n \to R$ be a sequence of lower semicontinuous functions. Suppose that, for each $x \in R^n$, there exists a constant M_x such that $f_n(x) \leq M_x$ for all n. Prove that there is a nonempty open set $O \subset R^n$ and a constant M such that, for all n,

$$f_n(x) \leq M, \ \forall \, x \in O.$$

5. Let X, Y be Banach spaces and let $T : X \mapsto Y$ be a bounded linear operator. If a sequence $\{x_n\} \subset X$ is weakly convergent, prove that the same is true for $\{Tx_n\}$.

6. Let S be the set of of all step functions on $[0, 1]$ with rational range and rational partition points.

(a) Show that the closure of S in $L_\infty[0, 1]$ contains $C[0, 1]$.

(b) Show that the S is dense in $L_1[0, 1]$.

7. An ordering on a topological space is an order compatible with the topology if when $x < y$, there are disjoint neighborhoods A of x and B of y such that for all $z \in A$ one has $z < y$ and for all $w \in B$ one has $x < w$.

(a) Show that the unit circle cannot be well ordered. (Hint: One proof follows the lines of the proof of the intermediate value theorem.)

(b) Show that you cannot find a well ordering of the plane R^2 that is compatible with the Euclidean distance in the plane.