2013 Analysis Preliminary Examination Makeup

Select 5 from the following 7 problems

1. Let $f_n : R \mapsto R$ be a sequence of functions that converges to f everywhere. Prove or disprove:

$$\{x : f(x) \le a\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : f_n(x) < a + \frac{1}{m}\}.$$

2. (a) State the Lebesgue bounded convergence theorem.
(b) Suppose that $f : [0,1] \to R$ is a continuous function. Prove $\lim_{n\to\infty} \int_0^1 f(x^n)dx$ exists and evaluate the limit.
(c) Does the limit in (b) always exist if f is only integrable.

3. Let X be a Banach space and let $x_n \in X$ be a weakly convergent sequence. Prove or disprove: $\|x_n\|$ is bounded.

4. (a) Give the definition that a function f is absolutely continuous on $[0,1]$.
(b) Prove or disprove: if f and g are absolutely continuous then $f \cdot g$ is absolutely continuous.
(c) Prove or disprove: if f is absolutely continuous and g is continuous then $f \cdot g$ is absolutely continuous.

5. (a) Prove or disprove: $C[a,b]$ is dense in $L^1[a,b]$.
(b) Prove or disprove: $C[a,b]$ is dense in $L^\infty[a,b]$.

6. (a) State Fubini's Theorem for L^1 functions on measure spaces. Be sure to state the hypotheses and conclusions fully and precisely.
(b) Let $f(m,n) \in R$ for all positive integers m, n. Suppose that

$$\sum_{m=1}^{\infty} |f(m,n)| \le \frac{1}{n^2}$$

for each positive integer n. Using your statement in (a), prove that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n).$$

7. Show that a sequence of measurable functions f_n on R^n converges to 0 in measure if and only if

$$\lim_{n\to\infty} \int \frac{|f_n|}{1+|f_n|} = 0.$$