Topology Preliminary Exam

September 1, 2009

There are ten problems. You may choose to do any seven of them.

Problem 1 *Let X be a connected space. A covering of X is a fibration $\widetilde{X} \xrightarrow{p} X$ so that the fibres are discrete spaces.*

> *(a) Explain how $\pi_1(X)$ acts on $\widetilde{X}$. Prove your assertions.*
>
> *(b) Calculate $\pi_1(\mathbb{R}P^2)$, $\pi_2(\mathbb{R}P^2)$, and $\pi_3(\mathbb{R}P^2)$.*

Problem 2 *Let $F \to E \xrightarrow{p} B$ be a fibre sequence.*

> *(a) Show there is a map $g : E/F \to B$ so that $E \to E/F \xrightarrow{g} B$ is p.*
>
> *(b) Determine the homotopy fibre of g.*

Problem 3 *Determine, with justification, whether or not $S^2 \times S^4$ is homotopy equivalent to the complex projective space $\mathbb{C}P^3$.*

Problem 4 *Given is a diagram of R-modules and homomorphisms, with all rectangles commutative,*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ B_1 & \xrightarrow[g_1]{} & B_2 & \xrightarrow[g_2]{} & B_3 & \xrightarrow[g_3]{} & B_4 & \xrightarrow[g_4]{} & B_5 \end{array}$$

such that the rows are exact (at locations labeled with subscripts 2, 3, 4). If α is epic and β, δ are both monic, prove that γ is monic.

Problem 5 *For a fixed $n \in \mathbb{N}$, compute the integral cohomology of the union of three n-disks along their common boundaries.*

Problem 6 *Let X be an $(n-1)$-connected space. Define the Hurewicz homomorphism $\mathcal{H} : \pi_n(X) \to H_n(X; \mathbb{Z})$. Show $\mathcal{H}$ is an isomorphism, if $n \geq 2$. What happens if $n = 1$?*

Problem 7 *Let D^n denote the closed n-ball. Prove the Brouwer fixed-point theorem that any continuous map $f : D^n \to D^n$ has a fixed point.*

Problem 8 *Suppose G is a connected topological group and that N is a discrete normal subgroup of G. Show that N is contained in the center of G.*

Problem 9 *(a) For a space X, determine the homotopy colimit of the diagram*

$$\begin{array}{ccccc} * & \longleftarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * & \longrightarrow & *. \end{array}$$

(b) What is the dual statement?

Problem 10 *Let $\nabla : X \vee X \to X$ be the folding map. Determine the homotopy fiber of ∇.*