Topology Preliminary Exam

October 23, 2010

There are ten problems. You may choose to do any seven of them.

Problem 1 *Show that the unitary group $U(n+1)$ acts transitively on the sphere S^{2n+1}, with stabilizer subgroup $U(n)$. Then use this to determine, with justification, the relationship between i and n so that*

$$\pi_i(U(n)) \cong \pi_i(U(n+1)) \cong \pi_i(U(n+2)) \cong \cdots .$$

Problem 2

Let $p : (E, e_0) \to (X, x_0)$ be a covering space, with E connected and locally path-connected, and with group of covering transformations G. Let N be the normalizer of $p_*(\pi_1(E, e_0))$ in $\pi_1(X, x_0)$. Define a homomorphism from N to G that fits into a short exact sequence

$$0 \to p_*(\pi_1(E, e_0)) \to N \to G \to 0$$

of groups. Justify your claims.

Problem 3 *Let D^n denote the closed n-ball. Prove the Brouwer fixed-point theorem that any continuous map $f : D^n \to D^n$ has a fixed point.*

Problem 4 *Let $A \to B \to C$ be a cofibration sequence. For any X, show that the induced sequence $[C, X] \to [B, X] \to [A, X]$ is exact.*

Problem 5 *Let X be any space, and let $A \subseteq X$. Let*

$$E = \{(x, t) \in X \times I \mid x \in A \text{ if } t = 0\}$$

Show that the projection $p : E \to X$ given by $p(x, t) = x$ is a fibration.

Problem 6 *Let $f : X \to Y$. Then we can form the cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} C_g \xrightarrow{i} C_h$$

Show that $C_g \simeq \Sigma X$ and $C_h \simeq \Sigma Y$.

Problem 7 *If* $f : S^n \to S^n$*, then the induced a map on homotopy groups may be identified with a homomorphism* $\mathbb{Z} \to \mathbb{Z}$*. Such a homomorphism must be given by multiplication by an integer* d*, which is called the* **degree** *of* f*.*

(a) *Show that* $\deg(f \circ g) = \deg(f) \cdot \deg(g)$*.*

(b) *Show that reflection in any hyperplane defines a map* $r : S^n \to S^n$ *of degree* -1*.*

(c) *Determine the degree of the antipodal map* $x \mapsto -x$*.*

(d) *Show that if* $f : S^n \to S^n$ *has no fixed points, then* n *is odd.*

Problem 8 *Let* K *be the Klein bottle.*

(a) *Express* K *as the cofiber of a map* κ *between (wedges of) copies of* S^1*. Describe the map explicitly.*

(b) *Identify* ΣK *as a wedge sum.*

(c) *Determine the integral homology groups of* ΣK*.*

Problem 9 *Let* $f : S^1 \times D^2 \to S^3$ *be an embedding,* $\Sigma = f(S^1 \times \{0\})$ *and* $V = S^3 - \Sigma$ *(*Σ *is a knot with a tubular neighborhood). Use the Mayer-Vietoris sequence to calculate* $H_1(V; \mathbb{Z})$*.*

Problem 10 *Let* X *be a topological space obtained as the quotient space of the sphere* S^2 *under the equivalence relation* $x \sim -x$*, for* x *in the equatorial circle. Describe a CW-complex whose underlying space is* X*. Compute the homology* $H_*(X; \mathbb{Z})$*.*