Algebra Preliminary Exam
August 16, 2001

Instructions: Do all seven problems. You will have four hours for this exam.

1. (a) Give statements of the (first) three isomorphism theorems for groups.
   (b) (Goursat's Lemma) Let G_1 and G_2 be groups, and let H be a subgroup of $G_1 \times G_2$ such that the two projections $p_i : H \to G_i$ are surjective. Let N_1 be the kernel of p_2, and let N_2 be the kernel of p_1. One can identify N_1 and N_2 with normal subgroups of G_1 and G_2 respectively (check!). Prove that $G_1/N_1 \cong G_2/N_2$.

2. Let G be a simple group of order 60. Show that the action of G by conjugation on its set of Sylow subgroups gives an imbedding $G \hookrightarrow A_6$ of G into the alternating group on 6 letters.

3. Let R be a ring and M a (left) $R-$module.
   (a) Prove that $\text{End}_R(M) := \text{Hom}_R(M, M)$ is a ring.
   (b) (Schur's Lemma) Suppose M is a simple $R-$module (recall that M is *simple* if it is nonzero and has no nontrivial submodules). Prove that $\text{End}_R(M)$ is a division ring.

4. Observe that $\mathbb{Q}$ under addition is a module over the integers $\mathbb{Z}$.
   (a) Show $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.
   (b) Show $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

5. Let R be an integral domain.
   (a) Define what it means for a nonzero element p of R to be prime, and what it means for p to be irreducible.
   (b) Mark the following statements TRUE or FALSE.
      (i) If p is prime then p is irreducible.
      (ii) If p is irreducible then p is prime.
   (c) Of the items in (b), prove those which you marked TRUE.
   (d) Complete the following statement:

      An ideal P in R is prime if and only if R/P is .

      (Recall that, by assumption, a prime ideal is proper.)
   (e) Prove your assertion in (d).
   (f) Show that if R is a principal ideal domain, then $p \in R$ is prime if and only if it is irreducible.

6. Let K/F be a finite separable field extension of prime degree p. Suppose $K = F(\theta)$.
   (a) Show that θ has exactly p distinct Galois conjugates in some algebraic closure of K.
   (b) Let $\theta = \theta_1, \theta_2, \ldots, \theta_p$ be the Galois conjugates of θ . Show that if $\theta_2 \in K$ then K is Galois, with cyclic Galois group.

7. Let ζ be a primitive 12th root of unity.
   (a) Find all the automorphisms of the field extension $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (b) Find the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (c) Find all intermediary field extensions between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$.