Algebra Preliminary Exam
August 21, 2003

Instructions: Do all seven problems. You will have four hours for this exam.

1. Let M be a monoid. (Recall that a *monoid* M is a (non-empty) set, closed under an associative binary operation which has a two-sided identity.)
   (a) Let $F(M)$ be the set of all functions from M into M. Prove that $F(M)$ is a monoid under composition of functions. left
   (b) Prove that M acts on M by right multiplication.
   (c) Prove that M is isomorphic to a submonoid of $F(M)$.
   (d) Prove that M is a group if and only if M acts transitively on M by right multiplication.

2. Let P be a nontrivial, finite p-group, with center $Z(P)$. If N is a nontrivial, normal subgroup of P, prove that $N \cap Z(P)$ is nontrivial.

3. Let R be a commutative ring with identity and let M be a maximal ideal of R.
   (a) Show $M[x]$ is an ideal of $R[x]$ (for x an indeterminate).
   (b) Show $M[x]$ is a prime ideal, but not a maximal ideal.
   (c) Find a maximal ideal of $R[x]$ that contains $M[x]$.

4. Let $p(x) = x^5 - 2$, K the splitting field (in $\mathbb{C}$) of $p(x)$ over $\mathbb{Q}$, and let $G = Gal(K/\mathbb{Q})$.
   (a) Find the order of G and a set of generators. Is G abelian?
   (b) Find the intermediate field $\mathbb{Q} \subset E \subset K$ such that E is not a normal extension of $\mathbb{Q}$. Prove this in two ways: using the definition and the Fundamental Theorem of Galois Theory.
   (c) Find an extension F of $\mathbb{Q}$ of degree two such that $\mathbb{Q} \subset F \subset K$, and a set of generators of $Gal(\cancel{F/K})$.
   K/F

5. Let V and W be modules over a ring F, and let $T \in \text{Hom}_F(V, W)$. Let $Z = \{(v, T(v)) \mid v \in V\}$.
   (a) Show that $V \times W$ is a module over F.
   (b) Show that Z is a submodule of $V \times W$.
   (c) If F is a field, V is a vector space of dimension n, W is a vector space of dimension m and T is of rank r, what is the dimension of Z? Verify your claim.

6. Let R be a commutative ring with 1. Recall that R is *local* if R has a unique maximal ideal.
   (a) Show that R is local if and only if the set of all nonunits of R is an ideal in R.
   (b) Suppose R is local, S is a nonzero commutative ring with 1, and $f : R \to S$ is a surjective ring homomorphism. Show that S is a local ring.

7. Let G be a group, with center $Z(G)$.
   (a) Suppose $G/Z(G)$ is cyclic. Prove that G is abelian.
   (b) Suppose G has order 441. Prove that G is solvable. (Prove every claim/result you utilize in your solution, except for named theorems.)