Algebra Preliminary Exam
August 15, 2005

1. (a) Show that every group of order 66 has a subgroup of order 33.
   (b) Classify (up to isomorphism) all groups of order 66. (Hint: There are four of them.)

2. Let R be the ring $\mathbb{Z}/36\mathbb{Z}$, under the usual operations of addition and multiplication.
   (a) Classify all maximal ideals of R.
   (b) Classify all prime ideals of R.

3. Let S_4 be the symmetric group on 4 letters.
   (a) Show the center $Z(S_4) = 1$. Conclude the inner automorphism group of S_4 is isomorphic to S_4.
   (b) Find a non-trivial homomorphism from $Aut(S_4)$ to S_4. (Hint: Consider the set of Sylow 3-subgroups of S_4.)
   (c) Show $Aut(S_4) \cong S_4$.

4. Let R be the quotient $\mathbb{C}[x]/I$ where I is a proper ideal of $\mathbb{C}[x]$, and M a maximal ideal of R. Prove that the quotient R/M is isomorphic (as a ring) to $\mathbb{C}$.

5. Recall that a Euclidean domain is an integral domain D with a norm function N from D into the non-negative integers such that $N(0) = 0$, and such that there is a Division Algorithm. Let D be a Euclidean domain, and assume that the norm function satisfies $N(a) \leq N(ab)$ for all nonzero elements of D.
   (a) Show that $N(-a) = N(a)$ for all $a \in D$.
   (b) Show that if N satisfies $N(a+b) \leq max\{N(a), N(b)\}$ for all a and b in D then the quotient and remainder in the Division Algorithm are unique.
   (c) Show that D is a principal ideal domain.
   (d) Prove or disprove: $I = \{a \in D : N(a) > N(1_D)\}$ is an ideal in D.

6. Suppose M, N are left R-modules, for R an associative ring with identity.

   (a) Suppose $\mu : M \to N$ is an R-module homomorphism and $M \xrightarrow{\mu} N \to 0$ is split exact, with splitting map $\rho : N \to M$. Prove that $M \cong \operatorname{Im} \rho \oplus \operatorname{Ker} \mu \cong N \oplus \operatorname{Ker} \mu$.

   (b) Suppose P is an R-module such that the exact sequence $M \to P \to 0$ splits for every R-module M. Show that P is a direct summand of a free R-module F. (HINT: Use (a).)

7. Let k be an algebraically closed field. Describe in detail the relationship between the Jordan canonical form of a matrix over k and the Fundamental Theorem of Finitely Generated Modules over a Principal Ideal Domain.

8. Let K be an extension field of a field F. We say that two intermediate fields E and L are *conjugate* if there exists $\sigma \in Gal(K/F)$ such that $\sigma(E) = L$.

   (a) Show that E and L are conjugate if and only if $Gal(K/E)$ and $Gal(K/L)$ are conjugate subgroups of $Gal(K/F)$.

   (b) Give an example of fields K, F, E, and L as in (a) such that E and L are conjugate but distinct.