Algebra Preliminary Exam
December 17, 2003

Instructions: Do all nine problems. You will have six hours for this exam.

1. Let P be a p-Sylow subgroup of a group G and let H be a subgroup of G containing P.
   (a) Show that if $P \lhd H$ and $H \lhd G$, then $P \lhd G$.
   (b) Show that $N_G(N_G(P)) = N_G(P)$.

2. Let R be a commutative ring with 1. If M is an R-module, let $\mathrm{End}_R(M)$ denote the set of R-module homomorphisms from M into M.
   (a) Show that $\mathrm{End}_R(M)$ is a ring (not necessarily commutative) with 1. (You must define the operations on $\mathrm{End}_R(M)$.)
   (b) Show that $\mathrm{End}_R(R) \cong R$ as rings.
   (c) Generalize (b) to $\mathrm{End}_R(R/I)$, where I is an ideal in R.

3. Let R be a commutative ring and I an ideal in R. The **radical $\sqrt{I}$ of I** is defined by
$$\sqrt{I} = \{r \in R \,|\, r^m \in I \text{ for some integer } m \geq 1\}.$$
   An ideal J of R is called a **radical ideal** (or just **radical**) if $J = \sqrt{J}$.
   (a) Prove that $\sqrt{I}$ is an ideal in R.
   (b) Prove that every prime ideal in R is radical.
   (c) Prove that $\sqrt{I}$ is radical.
   (d) Suppose R is not the zero ring. Prove that I is radical if and only if R/I has no nilpotent elements.

4. Let K be the splitting field of $x^6 - 25$ over $\mathbb{Q}$. Determine $Gal(K/Q)$. Explicitly determine all subfields of K, giving generators over $\mathbb{Q}$. Indicate which are Galois over $\mathbb{Q}$.

5. Let K be an extension field of a field F, and suppose $\alpha \in K$ is algebraic over F. Prove that $F(\alpha) \cong F[x]/(f(x))$, where $f(x)$ is an irreducible, monic polynomial of degree $n \geq 1$ satisfying $f(\alpha) = 0$. (NOTE: You may not presume the existence of such an $f(x) \in F[x]$.)

6. Let R and S be commutative rings with identity, and let M be a module for R.
   (a) State and prove a condition on M in order for M to be cyclic. (Recall that a module is *cyclic* if it is generated by a single element.)
   (b) Let $\psi : R \to S$ be a surjective ring homomorphism. Prove that S is a cyclic R-module.
   (c) Now give an example of a ring R and an R-module M for which M is not cyclic.

7. For four subgroups A, B, A', B' of a group G, let $A' \lhd A$ and $B' \lhd B$.
   (a) Show that $A' \cap B \lhd A \cap B$, and that $A' \lhd (A \cap B)A'$.
   (b) Show that $(A' \cap B)(A \cap B') \lhd (A \cap B)$.
   (c) Show that $(A \cap B')A' \lhd (A \cap B)A'$. (HINT: Define a homomorphism
$$\psi : A \cap B \to (A \cap B)A'/A',$$
   and then consider $\psi((A \cap B')(A' \cap B))$.)