Western Michigan University
Abstract Algebra Preliminary Exam
August 27, 2007

Instructions. Please write clearly, completely, and legibly. You have 6 hours to work this exam, starting at 9:00 a.m. and finishing at 3:00 p.m.

1. Prove that a group of order 48 must have a normal subgroup of order 8 or 16.

2. (a) Define both of the terms "finite extension" and "algebraic extension" in the context of field theory. (b) Does either imply the other? Prove your assertion(s).

3. Suppose $F \subset K \subset L$ are fields, $\theta \in L$, and $p \in F[x]$ is the minimal polynomial for θ. Prove that $K \otimes_F F(\theta)$ and $K[x]/(p)$ are isomorphic as algebras over K.

4. (a) If R is an integral domain with only a finite number n of ideals, show R is a field, and give an upper bound for n. (b) Prove that an integral domain has the descending chain condition on ideals if and only if it is a field.

5. Let K be the splitting field of $x^6 - 25$ over $\mathbb{Q}$. (a) As explicitly as possible, describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. (At the very least, describe the structure of the group in terms of more familiar groups and give a set of generators.) (b) Determine which subfields of K are Galois over $\mathbb{Q}$.

6. Throughout this problem, G is a finite group and all vector spaces are finite-dimensional. (a) Prove that G has a faithful representation. (In other words, explain why there is at least one one-to-one homomorphism $G \to GL(V)$, where V is a vector space.) (b) Prove that there is a bijective correspondence between modules over the group algebra $\mathbb{C}G$ and finite-dimensional representations of G.