Western Michigan University
Abstract Algebra Preliminary Exam
September 1, 2009

Instructions. Please write clearly, completely, and legibly. **Please put EACH problem on a separate piece of paper, with your name written at the top.** You have 6 hours to work this exam, starting at 9:00 a.m. and finishing at 3:00 p.m. Submit your solutions and this copy of your exam to the proctor (Dr. Hodge, before 1:30, Dr. Paul, after 1:30).

1. Let G be a finite group. Define the **dual** G^* of G to be the set of homomorphisms from G into the multiplicative group $\mathbb{C}^*$.
   (a) Define a natural group structure on G^*.
   (b) Show that if G and H are finite groups then $(G \times H)^* \cong G^* \times H^*$.
   (c) Show that if G is cyclic then $G^* \cong G$.
   (d) Show that if G is finite abelian then $G^* \cong G$.
2. Suppose A is an $n \times n$ matrix over $\mathbb{R}$ such that $A^2 = A$. Prove that the trace of A is equal to the rank of A. (Hint: Consider the eigenvalues of A.)
3. (a) Give an example of a ring R for which $R[x]$ is a Euclidean domain. (Remember to justify your answer.)
   (b) Give an example of a ring R for which $R[x]$ is NOT a Euclidean domain. (Remember to justify your answer.)
4. Show that a simple group of order 60 is isomorphic to A_5.
5. Suppose R is a ring. Recall the **annihilator** of a left R-module M is $Ann_R(M) = \{a \in R \,|\, aM = 0\}$. Suppose M_1 and M_2 are submodules of M.
   (a) Show that $Ann_R(M_1 + M_2) = Ann_R(M_1) \cap Ann_R(M_2)$.
   (b) Show furthermore that we have $Ann_R(M_1) + Ann_R(M_2) \subseteq Ann_R(M_1 \cap M_2)$.
   (c) Give an example to show that the inclusion in part (b) may be strict.
6. Suppose p is a prime number and q is a power of p. If F is a field of order q, prove, in detail, that the map $a \mapsto a^p$ for all $a \in F$, is an automorphism of F.
7. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 3$, and let K be a splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that $Gal(K/Q) \cong S_n$ (the symmetric group on n letters).
   (a) Show that f is irreducible over $\mathbb{Q}$.
   (b) If α is a root of f, show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.
   (c) If $n \geq 4$, show that $\alpha^n \in \mathbb{Q}$.