WMU Algebra Preliminary Exam

September 1, 2010

This exam is closed book, closed notes. There are 8 problems. You have six hours for this exam. Please put each problem on a separate sheet of paper, with your name at the top.

1. Prove that there exists $n \in \{60, 149, 200, 1092, 1365, 2907, 6545\}$ such that no group of order n is simple.

2. Let $\omega = e^{2\pi i/5}$.

   (a) If possible, find a field $\mathbb{F} \subset \mathbb{Q}(\omega)$ such that $[\mathbb{F}(\omega) : \mathbb{F}] = 2$. If not possible, explain why not.

   (b) If possible, find a field $\mathbb{F} \subset \mathbb{Q}(\omega)$ such that $[\mathbb{F}(\omega) : \mathbb{F}] = 3$. If not possible, explain why not.

3. Let Ω be a set, and G a subgroup of the group $Sym(\Omega)$ of permutations of Ω. Let $\omega \in \Omega$, and let G_ω denote the stabilizer of ω in G.

   (a) If H is a subgroup of G such that H acts transitively on Ω, show that $G = HG_\omega$.

   (b) If G acts transitively on Ω, show that $G/G_\omega \simeq \Omega$ as G-sets.

4. Suppose R is a commutative ring with 1. Whenever we say "R-module" we mean "left R-module". Set $J(R)$ to be the Jacobson radical of R; one can define it as

$$J(R) = \bigcap_{I \text{ maximal ideal in } R} I.$$

   (a) Prove that $J(R)$ is an ideal in R.

   (b) Show that

$$J(R) = \bigcap_{S \text{ simple } R\text{-module}} Ann(S),$$

   where for any R-module M, $Ann(M)$ is the annihilator of M in R.

   (c) Prove that $J(R/J(R)) = 0$.

5. Suppose that A is a ring with 1 that satisfies the property that $x^2 = x$ for all $x \in A$. Show that:

   (a) $2x = 0$ for all $x \in A$;

   (b) A is a commutative ring;

   (c) every prime ideal $\mathfrak{P}$ is maximal, and $A/\mathfrak{P}$ is the field of two elements.

6. Let V be a vector space of dimension n over a field F. If A is an endomorphism of V, then A induces an endomorphism ϕ_A on the vector space of all endomorphisms of V by sending an endomorphism B to AB. What possible dimensions can $\ker \phi_A$ attain?

7. Let p, $q \in \mathbb{Z}$ be primes.

   (a) Show that if $p \neq q$ then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = 0$.

   (b) Give an explicit $\mathbb{Z}$-module isomorphism $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}/p\mathbb{Z}$.

8. Let E be an extension of a field F with $[E : F] = 2$, and characteristic of F not equal 2. Prove that there exists $x \in E$ such that $x \notin F$, but $x^2 \in F$.