WMU Algebra Preliminary Exam
August 26, 2011

This exam is closed book, closed notes. There are two groups of five problems each. **You must choose four problems each from sections A and B.** If you attempt all five problems from one section, you must indicate clearly which four problems should be graded. You have six hours for this exam. Please put each problem on a separate sheet of paper, with your name at the top.

Problems A

A1 Let G be a group with H and K subgroups of G. Suppose further that H is normal in G, $G = HK$, and $H \cap K = \{1\}$. Show that G is isomorphic to a semidirect product $H \rtimes_\theta K$. Be sure to identify θ.

A2 Let G be a finite group, and let H be a subgroup of G.

(a) Let L be the set of left cosets of H in G. The group G acts on L by left multiplication, thereby defining a homomorphism $\pi : G \longrightarrow Perm(L)$. Show that $\ker(\pi) \subset H$. (Here, $Perm(L)$ is the group of permutations of the set L.)

(b) Suppose that the index of H in G is the smallest prime number p dividing the order of G. Prove that H is a normal subgroup of G.

A3 Let G be a group of order 105. Show that if G has a unique subgroup of order 3 then G is abelian.

A4 Let ℓ be an odd positive integer.

(a) Show that 2 is a unit in $\mathbb{Z}/\ell\mathbb{Z}$.

(b) Show that there exists a positive integer k such that $2^k \equiv 1 \bmod \ell$.

(c) Show that there exists a finite extension field F of the binary field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ such that all ℓ'th roots of unity exist in F.

(d) Determine the smallest such extension F when $\ell = 11$.

A5 Let $f(x) = x^4 + 1$.

(a) Let L be a splitting field (with characteristic $\neq 2$) for $f(x)$. Let G be the subgroup of $L^\times$ (the multiplicative group of units) generated by the roots of $f(x)$. Identify the (finite) group G. (What is its order and isomorphism class?)

(b) Let K be a subfield of L over which $f(x)$ is irreducible. Determine the Galois group of L over K, writing out the multiplication table, and explain all the symbols you are using.

Problems B

B1 Let A be a complex $n \times n$ matrix that is diagonalizable. Show that there exists a polynomial $p(x)$ with distinct roots such that $p(A) = \mathbf{0}_{n\times n}$.

B2 A *Lie Algebra* L is a vector space over a field k with a binary operation $[\cdot, \cdot]$ known as the *Lie bracket*. This operation is bilinear, satisfies for all $x \in L$, $[x, x] = 0$, and satisfies the *Jacobi identity*: for all $x, y, z \in L$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Let M be a vector subspace of L. What further properties must M have in order for the quotient space L/M to inherit the structure of a Lie algebra from L?

B3 Suppose R is a commutative ring with 1, P is a prime ideal of R, and S is the set-theoretic difference $S = R - P$.

- **(a)** Show that S is closed under multiplication, and does not contain 0 or any zero divisors.
- **(b)** Show that $S^{-1}RP$ is the unique maximal ideal in $S^{-1}R$.

B4 If R is a commutative ring and Q is an R-module, we call Q *injective* if it has the property that for all R-modules M and N and homomorphisms $f : M \longrightarrow Q$ and $\varphi : M \longrightarrow N$ such that φ is injective, there is a homomorphism $\overline{f} : N \longrightarrow Q$ such that $\overline{f} \circ \varphi = f$. Show that if F is a field and Q a finite dimensional vector space over F then Q is an injective F-module.

B5 If A is a ring with 1, show that

- **(a)** A acts (as an additive group) on A by right addition.
- **(b)** A acts (as a monoid under multiplication) on A by left multiplication. (Recall that a monoid is a set with an associative binary operation and a unit element. The definition of an action by a monoid is analogous to that of a group action.)
- **(c)** A^n is a free module over A of rank n.